

KNOTS

**From combinatorics of knot diagrams to combinatorial topology
based on knots**

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e-print: <http://arxiv.org/abs/math/0703096>

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Introduction

This book is about classical Knot Theory, that is, about the position of a circle (a knot) or of a number of disjoint circles (a link) in the space R^3 or in the sphere S^3 . We also venture into Knot Theory in general 3-dimensional manifolds.

The book has its predecessor in Lecture Notes on Knot Theory, which were published in Polish¹ in 1995 [P-18]. A rough translation of the Notes (by Jarek Wiśniewski) was ready by the summer of 1995. It differed from the Polish edition with the addition of the full proof of Reidemeister's theorem. While I couldn't find time to refine the translation and prepare the final manuscript, I was adding new material and rewriting existing chapters. In this way I created a new book based on the Polish Lecture Notes but expanded 3-fold. Only the first part of Chapter III (formerly Chapter II), on Conway's algebras is essentially unchanged from the Polish book (except new Subsection III.1.1 on Monoid of Conway algebras), and is based on preprints [P-1].

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SEE INTRODUCTION AND CHAPTER I OF THE BOOK.

¹The Polish edition was prepared for the "Knot Theory" mini-semester at the Stefan Banach Center, Warsaw, Poland, July-August, 1995.

Chapter III

Conway type invariants of links and Kauffman's method

III.1 Conway algebras

While considering quick methods of computing Alexander polynomial¹ (a classical invariant of links, compare Chapter IV for another approach to the Alexander polynomial), Conway [Co-1] suggested a normalized form of it (now called the Conway or Alexander-Conway polynomial) and he showed that the polynomial, $\Delta_L(z)$, satisfies the following two conditions:

- (i) (Initial condition) If T_1 is the trivial knot then $\Delta_{T_1}(z) = 1$.
- (ii) (Conway's skein relation) $\Delta_{L_+}(z) - \Delta_{L_-}(z) = z\Delta_{L_0}$, where L_+, L_- and L_0 are diagrams of oriented links which are identical except for the part presented in Fig. 1.1.

The conditions (i) and (ii) define the Conway polynomial (or, maybe more properly, Alexander-Conway polynomial) $\Delta_L(z)$ uniquely, see [Co-1, K-1, Gi, B-M]. Alexander used the variable t in his polynomial. For $z = \sqrt{t} - \frac{1}{\sqrt{t}}$ we obtain the normalized version of the Alexander polynomial. The skein

¹One can find this statement ironic because the classical Alexander method, which uses certain determinant (see Chapter IV), to compute the Alexander polynomial has polynomial time complexity, the method developed by Conway has an exponential time complexity.

relation has now the form:

$$(ii') \quad \Delta_{L_+} - \Delta_{L_-} = \left(\sqrt{t} - \frac{1}{\sqrt{t}}\right) \Delta_{L_0}.$$

In fact the un-normalized version of the formula (ii') was noted by J.W. Alexander in his original paper introducing the polynomial [Al-3], in 1928. Alexander polynomial was defined up to invertible elements, $\pm t^i$, in the ring of Laurent polynomials, $\mathbb{Z}[t^{\pm 1}]$, so the formula (ii) was not easily available for a computation of the polynomial.

In May of 1984 V. Jones, [Jo-1, Jo-2], showed that there exists an invariant V of links which is a Laurent polynomial with respect to the variable \sqrt{t} which satisfies the following conditions:

- (i) $V_{T_1}(t) = 1$,
- (ii) $\frac{1}{t}V_{L_+}(t) - tV_{L_-}(t) = \left(\sqrt{t} - \frac{1}{\sqrt{t}}\right)V_{L_0}(t)$.

These two examples of invariants were a base for an idea that there exists an invariant (of ambient isotopy) of oriented links which is a Laurent polynomial $P_L(x, y)$ of two variables² and which satisfies the following conditions:

- (i) $P_{T_1}(x, y) = 1$
- (ii) $xP_{L_+}(x, y) + yP_{L_-}(x, y) = P_{L_0}(x, y)$.

Indeed, such an invariant exists and it was discovered a few months after the Jones polynomial, in July-September of 1984, by four groups of mathematicians: R. Lickorish and K. Millett, J. Hoste, A. Ocneanu as well as by P. Freyd and D. Yetter [FYHLMO]. Independently, it was discovered in November-December³ of 1984 by J. Przytycki and P. Traczyk [P-T-1]). We call this polynomial the Homflypt or Jones-Conway polynomial⁴.

²Most of us started from a polynomial of 3-variables $P_L(x, y, z) \in \mathbb{Z}[x^{\pm 1}, y^{\pm 1}, z]$ and then assuming, with only partial justification that z can be assumed to be invertible, e.g. $z = 1$.

³In fact, I had to stop thinking for few days on the idea of the proof of existence of generalized Jones polynomial because I had to prepare, by the end of November of 1984, the syllabus for (an early version) of this book.

⁴HOMFLYPT (or, as I prefer to write: Homflypt) is the acronym after the initials of the inventors: Hoste, Ocneanu, Millett, Freyd, Lickorish, Yetter, Przytycki, and Traczyk. We note also some other names that are used for this invariant: FLYPMOTH, HOM-FLY, the generalized Jones polynomial, two variable Jones polynomial, twisted Alexander polynomial and skein-polynomial.

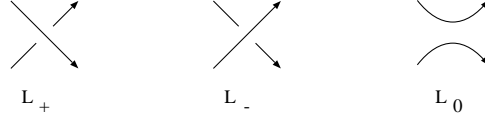


Fig. 1.1

Instead of looking for polynomial invariants of links related to Fig. 1.1 we can approach the problem from a more general point of view. Namely, we can look for universal invariants of links which have the following property: a given value of the invariant for L_+ and L_0 determines the value of the invariant for L_- , and similarly: if we know the value of the invariant for L_- and L_0 we can find its value for L_+ . The invariants with this property are called Conway type invariants. We will develop these ideas in the present chapter of the book which is based mainly on a joint work of Traczyk and the author [P-T-1, P-1].

Let us consider the following general situation involving an abstract algebra \mathcal{A} ; in our setting a set (called universe) A together with countable number of 0-argument operations (fixed elements) $a_1, a_2, \dots, a_n, \dots$ and two 2-argument operations $|$ and \star . We would like to construct an invariant w of oriented links with values in A which satisfies the following conditions:

$$\begin{aligned} w_{L_+} &= w_{L_-} | w_{L_0} \text{ and} \\ w_{L_-} &= w_{L_+} \star w_{L_0} \text{ and} \\ w_{T_n} &= a_n \end{aligned}$$

where T_n is a trivial link of n components.

The operation $|$ is meant to recover values of the invariant w for L_+ from its values for L_- and L_0 while the operation \star is supposed to recover values of w for L_- from its values for L_+ and L_0 .

Definition III.1.1 *We say that $\mathcal{A} = (A; a_1, a_2, \dots, |, \star)$ is a Conway algebra if the following conditions are satisfied:*

$$\left. \begin{array}{l} C1 \quad a_n | a_{n+1} = a_n \\ C2 \quad a_n \star a_{n+1} = a_n \end{array} \right\} \text{ initial values properties}$$

$$\begin{array}{lcl}
C3 & (a|b)|(c|d) & = (a|c)|(b|d) \\
C4 & (a|b) \star (c|d) & = (a \star c)|(b \star d) \\
C5 & (a \star b) \star (c \star d) & = (a \star c) \star (b \star d)
\end{array} \left. \vphantom{\begin{array}{lcl} C3 \\ C4 \\ C5 \end{array}} \right\} \text{transposition or entropy properties}$$

$$\begin{array}{lcl}
C6 & (a|b) \star b & = a \\
C7 & (a \star b)|b & = a
\end{array} \left. \vphantom{\begin{array}{lcl} C6 \\ C7 \end{array}} \right\} \text{inversion properties.}$$

We will prove the following theorem which is the main result of this chapter.

Theorem III.1.2 *For a given Conway algebra \mathcal{A} there exists a uniquely determined invariant of oriented links w which to any class L of ambient isotopy of links associates an element $w_L \in A$ and satisfies the following conditions:*

- $$\begin{array}{lcl}
(1) & w_{T_n} = a_n & - \text{initial conditions} \\
(2) & w_{L_+} = w_{L_-}|w_{L_0} \\
(3) & w_{L_-} = w_{L_+} \star w_{L_0}
\end{array} \left. \vphantom{\begin{array}{lcl} (2) \\ (3) \end{array}} \right\} - \text{Conway relations}$$

The theorem will be proved in the next section.

Now we briefly discuss geometric interpretation of conditions $C1 - C7$ in the definition of Conway algebra. Conditions $C1$ and $C2$ are reflecting relations between trivial links of n and $n + 1$ components. The diagrams of the links, which are in these relations, are pictured in Fig. 1.2.

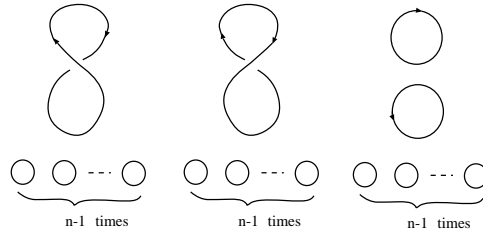


Fig. 1.2

Relations $C3$, $C4$ and $C5$ are obtained when we perform a calculation of a link invariant at two crossings of the diagram in different order. These relations will become apparent in Section 2.

Relations $C6$ and $C7$ illustrate the fact that we need the operations $|$ and \star to be opposite in some sense (see Lemma III.1.4(a) and Section III.2).

Before giving examples (models) of Conway algebras and proving the main theorem (in Section 2), we show some elementary properties of Conway algebras. In the definition of a Conway algebra we have introduced seven conditions. It was mainly because of aesthetic and practical reasons (we wanted to display the symmetry between the two relations). These conditions, however, are not independent one from another:

Lemma III.1.3 *There are the following dependencies between conditions $C1 - C7$ in the definition of the Conway algebra.*

- (a) $C1 \text{ and } C6 \Rightarrow C2$
- (b) $C2 \text{ and } C7 \Rightarrow C1$
- (c) $C6 \text{ and } C4 \Rightarrow C7$
- (d) $C7 \text{ and } C4 \Rightarrow C6$
- (e) $C6 \text{ and } C4 \Rightarrow C5$
- (f) $C7 \text{ and } C4 \Rightarrow C3$
- (g) $C5, C6 \text{ and } C7 \Rightarrow C4$
- (h) $C3, C6 \text{ and } C7 \Rightarrow C4$

We will prove, as examples, the implications (a), (c), (e), and (g)

- (a) $C1 \Leftrightarrow$
 $a_n | a_{n+1} = a_n \Rightarrow$
 $(a_n | a_{n+1}) \star a_{n+1} = a_n \star a_{n+1} \xRightarrow{C6}$
 $a_n = a_n \star a_{n+1} \Leftrightarrow$
 $C2.$
- (c) $C6 \Rightarrow$
 $(a | (b | a)) \star (b | a) = a \xLeftrightarrow{C4}$
 $(a \star b) | ((b | a) \star a) = a \xRightarrow{C6}$
 $(a \star b) | b = a \Leftrightarrow$
 $C7.$

(e) and (g):

$$\begin{aligned}
C5 &\Leftrightarrow (a \star b) \star (c \star d) = (a \star c) \star (b \star d) \\
&\quad \Downarrow C7 \quad \Uparrow C6 \\
(a \star b) &= ((a \star c) \star (b \star d)) | (c \star d) \\
&\quad C7 \Downarrow \text{ or } \Uparrow C6 \text{ and } C4 \\
[((a \star c) | c) \star ((b \star d) | d)] &= ((a \star c) \star (b \star d)) | (c \star d) \\
\text{we substitute } a = x | c &\Downarrow C6 \quad x = a \star c \quad \Uparrow \\
b = y | d &\Downarrow C6 \quad y = b \star d \quad \Uparrow \\
(x | c) \star (y | d) &= (x \star y) | (c \star d) \Leftrightarrow C4.
\end{aligned}$$

An interesting example of identity in Conway algebras is the following equality suggested by P. Traczyk:

$$((a|b) \star c) | d = (a | d) \star c | b$$

The following short proof of the identity is by C. Bowszyc:

$$\begin{aligned}
((a|b) \star c) | d &\stackrel{C6}{=} ((a|b) \star c) | ((d|e) \star e) \stackrel{C4}{=} ((a|b) | (d|e)) \star (c|e) \stackrel{C3}{=} \\
&((a|d) | (b|e)) \star (c|e) \stackrel{C4}{=} ((a|d) \star c) | ((b|e) \star e) \stackrel{C6}{=} ((a|d) \star c) | b.
\end{aligned}$$

Lemma III.1.4 (a) *Let \mathcal{A} be a Conway algebra. For any $b \in A$ let us consider a map $|_b : A \rightarrow A$ (respectively, $\star_b : A \rightarrow A$) defined by $|_b(a) = a | b$ (respectively, $\star_b(a) = a \star b$). Then $|_b$ and \star_b are bijections on A and one is the inverse of the other, that is: $|_b \cdot \star_b = \star_b \cdot |_b = \text{Id}$. If $| = \star$, we say that the Conway algebra is involutive.*

(b) *One can give equivalent definition of a Conway algebra using only one 2-argument operation, say $|$. The axioms are as follows:*

- (i) $a_n | a_{n+1} = a_n$,
- (ii) *The map $|_b : A \rightarrow A$ is a bijection,*
- (iii) $(a|b) | (c|d) = (a|c) | (b|d)$.

Proof: Lemma 1.4(a) follows from conditions C6 and C7.

Lemma 1.4(b) can be derived from Lemma 1.3. Let us show, for example that (iii) (i.e. C3) and (ii) (from which C6 and C7 follow immediately), imply C4. Namely,

By (iii) we have $((a \star b) | b) | ((c \star d) | d) = ((a \star b) | (c \star d)) | (b | d)$ and by C7 $((a \star b) | b) | ((c \star d) | d) = a \star c$, and by C6 $((a \star b) | (c \star d)) | (b | d) \star (b | d) = (a \star b) | (c \star d)$. Combining it together we obtain C4: $(a | c) \star (b | d) = (a \star b) | (c \star d)$. In fact assuming C6 and C7 we have that C3, C4 and C5 are equivalent.

For example $C4 \Rightarrow C3$:

By $C4$ we have $((a|b) * b)|((c|d) * d) = ((a|b)|(c|d)) * (b|d)$ or equivalently

$$(((a|b) * b)|((c|d) * d))|(b|d) = (((a|b)|(c|d)) * (b|d))|(b|d).$$

This formula reduces, by $C6$ and $C7$ to $(a|c)|(b|d) = (a|b)|(c|d)$, as needed.

□

Now, let us discuss some examples of Conway algebras.

Example III.1.5 (Number of components) *Let $A = N$ be the set of positive integers. We define $a_i = i$ and $i|j = i \star j = i$.*

Verification of conditions $C1 - C7$ is immediate (note that the first letter of each side of every relation is the same).

The invariant of a link defined by this algebra (it exists according to Theorem III.1.2) is equal to the number of components of the link.

Example III.1.6 *Let us set $A = \mathbb{Z}_3 = \{0, 1, 2\}$, $a_i \equiv i \pmod{3}$, and $a|b = a * b \equiv 1 - a - b \pmod{3}$. In other words $|$ and $*$ are both given by the following symmetric table:*

$ $	0	1	2
0	1	0	2
1	0	2	1
2	2	1	0

The invariant defined by this algebra distinguishes the trefoil knot from the trivial knot (see Fig. 1.3.).



Fig. 1.3; For the trivial knot the value of the invariant is: $a_1 = 1$. For the left-handed trefoil we have the value of the invariant: $a_1 * (a_2 * a_1) = 2$.

The direct generalization of Example III.1.6 can be obtained by taking $A = \mathbb{Z}_n$, $a_i \equiv i \pmod{n}$, and $a|b = a * b \equiv 2b - a - 2 \pmod{n}$.

Example III.1.7 Let us consider the universe $A = \{0, 1, 2, s\}$ with the distinguished elements $a_i \equiv i \pmod 3$. The operations $|$ and \star are given by the following tables (notice that $|$ is, but \star is not, a symmetric operation; for example $a_i \star a_{i+1} = a_i$, but $a_{i+1} \star a_i = s$).

$ $	0	1	2	s
0	s	0	2	1
1	0	s	1	2
2	2	1	s	0
s	1	2	0	s

\star	0	1	2	s
0	1	0	s	2
1	s	2	1	0
2	2	s	0	1
s	0	1	2	s

The invariant defined by this algebra distinguishes right-handed trefoil knot from the left-handed trefoil (see Fig. 1.4).

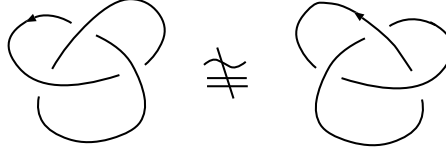


Fig. 1.4; For the left handed trefoil we have the value of the invariant: $a_1 \star (a_2 \star a_1) = 0$. For the right-handed trefoil we have the value of the invariant: $a_1 |(a_2 | a_1) = s$.

Using a computer, T. Przytycka found all Conway algebras with up to five elements. If we assume additionally that $a_1 = 1, a_2 = 2$, then (up to isomorphisms) we get⁵:

The number of elements in a Conway algebra	2	3	4	5
The number of isomorphism classes of Conway algebras	2	9	51	204

Example III.1.8 (Homflypt (Jones-Conway) polynomial) We set:

$$A = \mathbb{Z}[x^{\pm 1}, y^{\pm 1}], \quad a_1 = 1, \quad a_2 = x + y, \dots, a_i = (x + y)^{i-1}, \dots$$

The operations $|$ and \star are defined as follows: $w_2 | w_0 = w_1$ and $w_1 \star w_0 = w_2$, where the polynomials w_1, w_2, w_0 satisfy the equation:

III.1.9

$$xw_1 + yw_2 = w_0.$$

⁵E. Biendzio, student of P.Traczyk, found in her master degree thesis all 6-element Conway algebras [Bie].

The invariant of links defined by this algebra is the Jones-Conway (Homflypt) polynomial which we have mentioned at the beginning of this section. In particular, if we substitute $x = 1/z$ and $y = -1/z$, then we get the Conway polynomial, and after the substitution

$$x = \frac{-t}{\sqrt{t} - \frac{1}{\sqrt{t}}}, \quad y = \frac{1}{t} \frac{1}{\sqrt{t} - \frac{1}{\sqrt{t}}},$$

we obtain the Jones polynomial.

Now we shall prove that the algebra defined in Example III.1.8 is a Conway algebra.

First we note that conditions $C1$ and $C2$ follow from the identity

$$x(x+y)^{n-1} + y(x+y)^{n-1} = (x+y)^n.$$

Next, conditions $C6$ and $C7$ follow from the fact that the operations $|$ and \star were defined by the linear equation 1.9.

We prove the condition $C3$ and the conditions $C4$ and $C5$ follow by Lemma 1.3.

We get:

III.1.10

$$\begin{aligned} (a|b)|(c|d) &= \\ \frac{1}{x}((c|d) - y(a|b)) &= \\ \frac{1}{x}(\frac{1}{x}(d - yc) - y\frac{1}{x}(b - ya)) &= \\ \frac{1}{x^2}d - \frac{y}{x^2}c - \frac{y}{x^2}b + \frac{y^2}{x^2}a, \end{aligned}$$

and thus, because the coefficients of b and c are equal, it follows that we can interchange b and c in the formula, which proves the relation $C3$.

One may generalize the algebra from Example 1.8 by introducing a new variable z and considering instead of Equation 1.9 the equation

$$xw_1 + yw_2 = w_0 - z.$$

It turns out, however, that the invariant obtained this way is not stronger than the Jones-Conway polynomial (see Exercise III.3.43).

Example III.1.11 (Global linking number) *Let us set $A = N \times \mathbb{Z}$ and $a_i = (i, 0)$, and moreover*

$$(a, b) | (c, d) = \begin{cases} (a, b+1) & \text{if } a > c \\ (a, b) & \text{if } a \leq c \end{cases}$$

$$(a, b) \star (c, d) = \begin{cases} (a, b-1) & \text{if } a > c \\ (a, b) & \text{if } a \leq c \end{cases}$$

The invariant defined by this algebra is a pair, the first entry of which is the number of components of the link and the second entry is called the global linking number (or index) (see Exercise 1.12 and Chapter IV).

Notice that we have $(a, b) | (a, b) = (a, b) = (a, b) \star (a, b)$ that is idempotency condition holds, and that idempotency condition and Conway (entropy) relations C3 – C5 lead to distributivity (left and right distributivity). For example, if we put $b = d$ in C4 we get:

$$(x|y) \star (z|y) \stackrel{entr}{=} (x \star z) | (y \star y) \stackrel{idem}{=} (x \star z) | y; \text{ right distributivity of } | \text{ with respect to } \star.$$

Such magmas $(A; |)$, satisfying idempotency condition, invertibility, and right distributivity are called Quandles and the conditions reflect the Reidemeister moves [Joy]. If idempotency conditions are not assumed, these magmas are called racks (introduced by Conway and Wraith in 1959 [C-W]), and if only right self-distributivity is kept they are called right shelves, or just shelves (the word coined by Alissa Crans [Cr]; compare [P-40]).

Now we shall prove that the algebra from Example 1.11 is a Conway algebra.

The proof of conditions C1, C2, C6 and C7 is not hard. We will check condition C3 in more detail. Because of our definition of the operation $|$ in \mathcal{A} we get:

$$((a_1, a_2) | (b_1, b_2)) | ((c_1, c_2) | (d_1, d_2)) = \begin{cases} (a_1, a_2 + 2) & \text{if } a_1 > b_1 \text{ and } a_1 > c_1 \\ (a_1, a_2 + 1) & \text{if } a_1 > b_1 \text{ and } a_1 \leq c_1 \\ & \text{or } a_1 \leq b_1 \text{ and } a_1 > c_1 \\ (a_1, a_2) & \text{if } a_1 \leq b_1 \text{ and } a_1 \leq c_1 \end{cases}$$

Now, if we change the positions of b_1 and c_1 then the result will be the same. Therefore the relation $C3$ is satisfied.

The global linking number of an oriented link can be read directly from a diagram of a link. Namely, let us call a crossing of type \times positive and a crossing of type \times negative. We will write $\text{sgn}p = +$ or $-$ depending on whether the crossing p is positive or negative.

Exercise III.1.12 Suppose that D is a diagram of an oriented link. Let us define $\text{lk}(D) = \frac{1}{2} \sum \text{sgn}(p)$, where the sum is taken over all crossings p , between different components of the link. Show that $\text{lk}(D)$ is equal to the global linking number of the link.

Hint. Let us note that, if L_+, L_-, L_0 are diagrams of links as on the Fig. 1.1, then

$$\text{lk}(L_+) = \begin{cases} \text{lk}(L_-) + 1 & \text{if the crossing involves two different} \\ & \text{components of the link} \\ \text{lk}(L_+) & \text{if the crossing involves only one} \\ & \text{component of the link} \end{cases}$$

Moreover, note that the number of components of L_0 is smaller by one than the number of components of L_+ if the crossing involves two different components and it is bigger by one otherwise.

Exercise III.1.13 Consider the universe $A = N \times B$ for any set B . Let $f_{i,j} : B \rightarrow B$ be a bijection for any pair of natural numbers, and assume that for fixed i the functions commute (i.e. $f_{i,j}f_{i,k} = f_{i,k}f_{i,j}$). Check that $(A; |)$ with $(a_1, a_2)|(b_1, b_2) = (a_1, f_{a_1, b_1}(a_2))$ forms an entropic system with an invertible $|$. If we further assume that $f_{i, i+1} = \text{Id}$ then for a fixed $b_0 \in B$, $(A; a_1, a_2, \dots, |)$ with $a_i = (i, b_0)$ forms a Conway algebra.

We will write L_+^p, L_-^p and L_0^p instead of L_+, L_- and L_0 if we want the crossing point p to be explicitly specified.

Definition III.1.14 Let T be a binary tree of links, that is a binary tree each of whose vertices represents a link. We draw T with the root at the top. The root represents the given link L . The leaves in the bottom (vertices of degree one), represent trivial links. Moreover we assume that at each vertex which is not a leaf the situation is as follows:

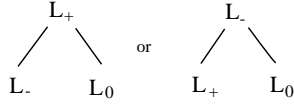


Fig. 1.5

The tree T yields, in a natural way, an associated tree with a_i at each of leaves and signs $+$ or $-$ at any other vertex. We call it a resolving tree of the link L (see Fig. 1.7 for an example); compare [F-M].

There exists a standard procedure for resolving a link. It will be described in the subsequent section and it will play an essential role in the proof of Theorem III.1.2.

Example III.1.15 Let L be the figure-eight knot represented by the diagram on Fig. 1.6a.

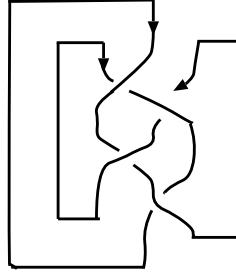


Fig. 1.6a

In order to determine w_L (i.e. the value of an invariant w associated to a Conway algebra \mathcal{A} on the link L) let us consider the following binary tree of links:

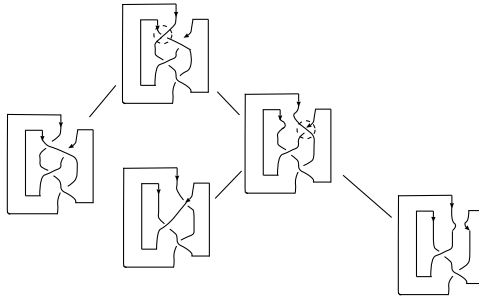


Fig. 1.6b

It can be easily seen that leaves of this tree represent trivial links and the branching vertices of the tree are related to the admissible operations on the diagram of the tree at each of the marked crossing points. The above tree defines the following resolving tree for the figure-eight knot:

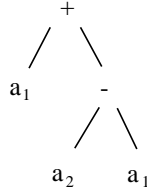


Fig. 1.7

Applying this tree we get $w_L = a_1|(a_2 \star a_1)$.

Exercise III.1.16

1. Show that the knot pictured on Fig. 1.5(b) is ambient isotopic to the one pictured on Fig. I:0.1.
2. Show that the figure-eight knot is ambient isotopic with its mirror image, that is

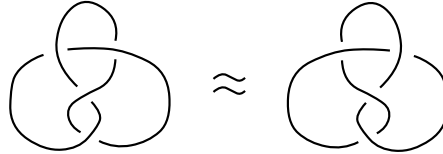


Fig. 1.8

3. Prove that the following identity holds in any Conway algebra

$$a_1 \star (a_2|a_1) = a_1|(a_2 \star a_1).$$

(This is an algebraic version of the property (2) of the figure-eight knot, c.f. Example III.1.15.)

4. Prove that if $*$ is an invertible binary operation and entropy property holds (that is C3, C6, and C7) and there is c such that $a * c = a$, then

$$(a * b)|a = a|(b * a).$$

5. Compute values of invariants from Examples 1.6, III.1.7, III.1.8 and III.1.11 for the figure eight knot.

Exercise III.1.17 Let K be a left handed trefoil knot and let \overline{K} be a right handed trefoil knot. Prove that $w_K = a_1 \star (a_2 \star a_1)$ and $w_{\overline{K}} = a_1 |(a_2 | a_1)$ (c.f. Fig. 1.4).

Exercise III.1.18 Show that the figure-eight knot is isotopic neither with the trefoil knot (left- and right-handed) nor with the trivial knot.

For a long boring evening we suggest the following exercise (c.f. Example III.1.7).

Exercise III.1.19 Prove that there does not exist any three element Conway algebra which distinguishes the right-handed trefoil knot from the left-handed trefoil knot.

III.1.1 Monoid of Conway algebras

Recall that a *magma* $(X; *)$ is a set X with a binary operation $* : X \times X \rightarrow X$. For any $b \in X$ the adjoint map $*_b : X \rightarrow X$, is defined by $*_b(a) = a * b$. Let $\text{Bin}(X)$ be the set of all binary operations on X .

Proposition III.1.20 [P-40] $\text{Bin}(X)$ is a monoid (i.e. semigroup with identity) with composition $*_1 *_2$ given by $a *_1 *_2 b = (a *_1 b) *_2 b$, and the identity $*_0$ being the right trivial operation, that is, $a *_0 b = a$ for any $a, b \in X$.

Proof: Associativity follows from the fact that adjoint maps $*_b$ compose in an associative way, $(*_3)_b((*_2)_b(*_1)_b) = ((*_3)_b(*_2)_b)(*_1)_b$; we can write directly: $a(*_1 *_2) *_3 b = ((a *_1 b) *_2 b) *_3 b = (a *_1 b)(*_2 *_3)b = a *_1 (*_2 *_3)b$. \square

The submonoid of $\text{Bin}(X)$ of all invertible elements in $\text{Bin}(X)$ is a group denoted by $\text{Bin}_{\text{inv}}(X)$. If $* \in \text{Bin}_{\text{inv}}(X)$ then $*^{-1}$ is usually denoted by $\bar{*}$, however in the case of Conway algebras the inverse of $*$ was denoted by $|$. Consider the set $a \circ c = \{b \in X \mid a * b = c\}$. If $*$ is invertible and $a \circ c$ has always exactly one element, we call the magma $(X; *)$ a quasigroup. If X is finite then the multiplication table for $*$ has a permutation in each row and column. Such a table is called a latin square – the objects studied for hundred of years (e.g. by Euler).

Probably Sushkevich⁶ in 1937 [Su] was the first to consider a binary operation satisfying entropic property $(a * b) * (c * d) = (a * b) * (c * d)$. He was motivated by the Burstin and Mayer paper of 1929, [Bu-Ma]. Soon after, Murdoch⁷ in [Mur] and Toyoda⁸ in a series of papers [Toy-1, Toy-2, Toy-3, Toy-4], have established main properties of such magmas and, in the case $(X; *)$ is a quasigroup, they proved the result name after them, Murdoch-Toyoda theorem (Theorem III.1.21). This result is very important in our search for invariants of Conway type. The name used by Toyoda was, associativity. The word *entropic property* which I use was coined in 1949 by I.M.H. Etherington⁹[Et]. Other names for the property are medial, alternation, bi-commutative, bisymmetric, commutative, surcommutative and abelian. The word entropic refers to inner turning [R-S-2].

Theorem III.1.21 [Mur, Toy-2] *If $(X; *)$ is an entropic quasigroup, then X has an abelian group structure such that $a * b = f(a) + g(b) + c$ where $f, g : X \rightarrow X$ are commuting group automorphisms (equivalently $(X; *)$ is a $\mathbb{Z}[x_f^{\pm 1}, x_g^{\pm 1}]$ -module).*

From this follows partially A. Sikora result [Si-2] for invariants coming from quasigroup Conway algebras (Sikora result is general: no invariant coming from a Conway algebra can distinguish links with the same Homflypt polynomial).

Definition III.1.22 (1) *We say that a subset $S \subset \text{Bin}(X)$ is an entropic set if all pairs of elements $*_{\alpha}, *_{\beta} \in S$ (we allow $*_{\alpha} = *_{\beta}$) satisfy*

⁶Anton Kazimirovich Sushkevich (1889-1961) a Ukrainian-Russian-Polish mathematician who spent most of his working life at Kharkov State University in the Ukraine. Ph.D. Voronezh State University 1922. In the 1920s, he embarked upon the first systematic study of semigroups, placing him at the very beginning of algebraic semigroup theory and, arguably, earning him the title of the world's first semigroup theorist. Owing to the political circumstances under which he lived, however, his work failed to find a wide audience during his lifetime [Hol].

⁷David Carruthers Murdoch, PhD at University of Toronto 1937, Then at Yale and University of British Columbia.

⁸Koshichi Toyoda, worked till 1945 at Harbin Polytechnic University, Mituhisa Takasaki was his assistant there. Both perished after Soviet occupation of Harbin in 1945.

⁹Ivor Malcolm Haddon Etherington (1908-1994), English mathematician who started his work in general relativity under the supervision of Prof. E. T. Whittaker, and later on moved into the area of non-associative algebras, where he made very important contributions in connection with genetics. He was actively involved in helping refugees to escape from Nazi Germany in the 1930s [Kra].

entropic property, that is:

$$(a *_\alpha b) *_\beta (c *_\alpha d) = (a *_\beta c) *_\alpha (b *_\beta d).$$

- (2) If $S \subset \text{Bin}(X)$ is an entropic set, and there are given elements a_1, a_2, \dots in X such that for any $*$ in S we have $a_i * a_{i+1} = a_i$ then we call S a pre-Conway set of operations.
- (3) If $S \subset \text{Bin}(X)$ is a pre-Conway set of invertible operations then we call S a Conway set of operations.

Proposition III.1.23 (i) If S is an entropic set and $*$ in S is invertible, then $S \cup \{\bar{*}\}$ is also an entropic set.

(ii) If S is an entropic set and $M(S)$ is the monoid generated by S then $M(S)$ is an entropic monoid.

(iii) If S is an entropic set of invertible operations and $G(S)$ is the group generated by S , then $G(S)$ is an entropic group.

Proof: (i) We have proven already that if $*$ is a self-entropic operation then $\{\bar{*}\}$ is also self-entropic (Lemma III:1.3 (e)(h)). We will show now, more generally, that if $*, *' \in \text{Bin}(X)$ and $*$ is invertible and entropic with respect to $*$ ', then $\bar{*}$ is entropic with respect to $*$ '. We start from

$$(a * b) *' (c * d) = (a *' c) * (b *' d)$$

Then we substitute $x = a * b$ and $y = c * d$, or, equivalently $a = x \bar{*} b$ and $c = y \bar{*} d$ to get

$$x *' y = ((x \bar{*} b) *' (y \bar{*} d)) * (b *' d) \quad \text{and eventually}$$

$$(x *' y) \bar{*} (b *' d) = (x \bar{*} b) *' (y \bar{*} d)$$

as needed.

(ii) We want to prove that any element of $M(S)$ is entropic with respect to any other element (including self-entropy of every element). To prove this it suffice to show that if $*$ is entropic with respect to $*_1$ and $*_2$ then it is also entropic with respect to $*_1 *_2$. We have

$$(a *_1 *_2 b) * (c *_1 *_2 d) \stackrel{def}{=} ((a *_1 b) *_2 b) * (c *_1 d) *_2 d \stackrel{entr}{=}$$

$$\begin{aligned}
& ((a *_1 b) * (c *_1 d)) *_2 (b * d) \stackrel{entr}{=} \\
& ((a * c) *_1 (b * d)) *_2 (b * d) \stackrel{def}{=} \\
& (a * c) *_1 *_2 (b * d), \quad \text{as needed.}
\end{aligned}$$

(iii) It follows directly from (i) and (ii). \square

Remark III.1.24 *Composition of entropic operations was considered in [R-S-1] where it was observed, in particular Theorem 246, that if S is an entropic set of idempotent operations then elements of S commute. Namely, we have $a *_1 *_2 b = (a *_1 b) *_2 b \stackrel{idem}{=} (a *_1 b) *_2 (b *_1 b) \stackrel{entr}{=} (a *_2 b) *_1 (b *_2 b) \stackrel{idem}{=} (a *_2 b) *_1 b = a *_2 *_1 b$ as needed.*

Proposition III.1.25 (1) *If $(A; *)$ satisfies entropic condition and idempotency conditions then it is (right and left) distributive.*

(2) *A right self-distributive quasigroup has all elements satisfying idempotent property.*

Proof: (1) We have $(a * b) * (c * b) \stackrel{entr}{=} (a * c) * (b * b) \stackrel{idem}{=} (a * c) * b$ (right self-distributivity), and

$(a * b) * (a * d) \stackrel{entr}{=} (a * a) * (b * d) \stackrel{idem}{=} a * (b * da)$ (left self-distributivity).

(2) We have $(a * a) * a = (a * a) * (a * a)$ so $a = a * a$. \square

A interesting example of an entropic algebra was proposed by D. Robinson in 1962 [Rob, R-S-2]. Consider a group of nilpotent class two¹⁰, that is $[[G, G], G] = 0$ where $[G, G]$ is generated by commutators $[g, h] = g^{-1}h^{-1}gh$. Then $(G; *)$ with $a * b = ba^{-1}b$ is an entropic magma (this operation was introduced by Bruck and called by him a core [Bruc-2]). Similarly if we consider on G a conjugacy operation $a * b = b^{-1}ab$; see Corollary III.1.27.

We prove the above observations in more general situation:

¹⁰The smallest nilpotent groups of class 2 are the quaternion group Q_8 , and the dihedral group $D_{2.4}$. Generally a finite group of a nilpotent class 2 are products of an abelian group and nonabelian p -groups of class two. In particular, any group of order p^3 . There are, up to isomorphism, two nonabelian groups of order p^3 : for $p = 2$ they are quaternion group Q_8 and dihedral group $D_{2.4}$; for $p > 3$ they are Heisenberg 3×3 matrices with \mathbb{Z}_p entries (semidirect product of \mathbb{Z}_{p^2} by \mathbb{Z}_p), and the group $\{a, b \mid a^{p^2} = b^p = 1, ab = ba^{p+1}\}$ (semidirect product of $\mathbb{Z}_p \oplus \mathbb{Z}_p$ by \mathbb{Z}_p). For two generator p -groups of nilpotency class two see [B-Ka, AMM]. An infinite example is the Heisenberg 3×3 matrices group with entries in \mathbb{Z} ; see [Du-Fo, Ma-Bi].

Proposition III.1.26 *Let $(X; *)$ be a magma with invertible $*$ and τ an involution on X . Conditions (1)-(3) below are equivalent and (4) is their consequence.*

$$(1) \quad (a * b) * c = (a * \tau(c)) * \tau(b);$$

$$(2) \quad (a * b) \bar{*} c = (a \bar{*} \tau(c)) * \tau(b)$$

$$(3) \quad (a \bar{*} b) \bar{*} c = (a \bar{*} \tau(c)) \bar{*} \tau(b)$$

$$(4) \quad (a * b) * (c * d) = (a * c) * (b * d); \text{ that is entropic property holds.}$$

Proof: (1)-(3) We show, for example, that (2) follows from (1) (other cases are similar). $(a * b) \bar{*} c = (a \bar{*} \tau(c)) * \tau(b)$ it is equivalent to $a * b = (((a \bar{*} \tau(c)) * \tau(b)) * c)$. The right side by (1) is $((a \bar{*} \tau(c)) * \tau(c)) * b = (a * b)$ as needed.

(4) We have

$$\begin{aligned} (a * b) * (c * d) &= (((a * b) \bar{*} d) * c) * d \stackrel{(2)}{=} \\ &(((a \bar{*} \tau(d)) * \tau(b)) * c) * d \stackrel{(1)}{=} (((a \bar{*} \tau(d)) * \tau(c)) * b) * d \stackrel{(2)}{=} \\ &(((a * c) \bar{*} d) * b) * d = (a * c) * (b * d) \text{ as needed.} \end{aligned}$$

□

Corollary III.1.27 *Let G be group of a nilpotent class 2, Then:*

- (i) *The core magma $(X; *)$ with $a * b = ba^{-1}b$ and $\tau(a) = a^{-1}$ satisfies conditions (1)-(4) of Proposition III.1.26.*
- (ii) *The conjugation magma $(X; *)$ with $a * b = b^{-1}ab$ and $\tau(a) = a$ satisfies conditions (1)-(4) of Proposition III.1.26.*

Proof: (i) In a core magma we have $(a * b) * c = (ba^{-1}b) * c = cb^{-1}ab^{-1}c$, and $(a * \tau(c)) * \tau(b) = (a * c^{-1}) * b^{-1} = b^{-1}cacb^{-1} = cb^{-1}[b^{-1}, c]a[c^{-1}, b]b^{-1}c$; Thus $(a * b) * c = (a * \tau(c)) * \tau(b)$ is equivalent to $a = [b^{-1}, c]a[c^{-1}, b]$ and further to $a^{-1}[b^{-1}, c]a[c^{-1}, b] = [a, [cb^{-1}]] [[cb^{-1}], c^{-1}b] = 1$ which holds in a group of class 2. Other conditions follow from Proposition III.1.26.

(ii) In a conjugation magma we have $(a * b) * c = (b^{-1}ab) * c = c^{-1}b^{-1}abc$ and $(a * \tau(c)) * \tau(b) = (a * c) * b = b^{-1}c^{-1}acb = c^{-1}b^{-1}[b^{-1}, c^{-1}]a[c^{-1}, b^{-1}]bc$. Thus $(a * b) * c = (a * \tau(c)) * \tau(b)$ is equivalent to $a = [b^{-1}, c^{-1}]a[c^{-1}, b^{-1}]$ and further to $a^{-1}[b^{-1}, c^{-1}]a[b^{-1}, c^{-1}]^{-1} = [a, [c^{-1}, b^{-1}]] \in [[G, G], G]$ as needed. Other conditions follow from Proposition III.1.26. □

The theory of homology of entropic magmas (analogous to that of homology of semigroups or shelves) is developed in [Ni-P].

III.2 Proof of the main theorem

Definition III.2.1 (*Descending diagram*).

Assume that L is an oriented link diagram of n components together with $b = (b_1 \dots, b_n)$ which are base points on L , each one chosen on a different link component of L (the base points are outside of the crossings of the diagram). Suppose that we move along the diagram L according to the orientation of L so that we start from b_1 , travel the first component ending in b_1 and then we travel the second component starting from b_2 , etc... Every crossing is traveled in that way twice. We say that the diagram L is descending with respect to b if each crossing that we meet on our way first is crossed by an overcrossing (a bridge). We say that L is ascending with respect to b if its mirror image, \bar{L} is descending with respect to b .

It is not hard to see that for any diagram L and any choice of base-points $b = (b_1 \dots, b_n)$ there exists a resolving tree such that its leaves are descending diagrams with respect to some appropriate choice of base points. Furthermore the diagram corresponding to the extreme left leaf has the same projection as L and is descending with respect to b .

To show this we apply induction with respect to the number k of crossings in a diagram. The diagrams with no crossings are already descending. Suppose that our claim is true for diagrams with fewer than $k > 0$ crossings. Now we can apply the following procedure for diagrams with k crossings. For any choice of base points we start walking along the diagram until we meet the first “bad” crossing p , i.e. the first crossing on our way which is entered for the first time at an underpass (tunnel). Then we begin to construct the tree changing the diagram at p . If, for example, $\text{sgn} p = +$, then we get

$$\begin{array}{c} L = L_+^p \\ \swarrow \quad \searrow \\ L_-^p \quad L_0^p \end{array}$$

Next, we can apply our inductive assumption on the diagram L_0^p and we continue the construction of the tree for the diagram L_-^p (i.e. we walk along the diagram looking for next “bad” crossings) until we eliminate all “bad” crossings. In L_-^p we use the same base points as in L . Later we will use similar reasoning several time so it is useful to formalize our double

induction: having a diagram with a basepoints we induct on lexicographically ordered pair: (the number of crossings of L , the number of “bad” crossings of L with respect to the chosen basepoint).

To prove Theorem III.1.2 we will construct a value function w on diagrams of links. In order to prove that w is an invariant of ambient isotopy of oriented links we will show that it is preserved by Reidemeister moves.

We will use the induction with respect to the number $\text{cr}(L)$ of crossing points in the diagram. For any $k \geq 0$ we will define a function w_k which assigns an element in A to each oriented diagram L with at most k crossings. Then we will define the function w by setting $w(L) = w_k(L)$, where $k \geq \text{cr}(L)$. Clearly, for this to work, the functions w_k must satisfy certain coherence conditions. Finally, the required properties of w will be obtained from these of w_k .

We begin by defining w_0 .

For a diagram L with n components and $\text{cr}(L) = 0$ we put:

III.2.2

$$w_0(L) = a_n$$

To define w_{k+1} and to prove its properties we will use induction several times. To avoid misunderstandings the following will be called the “Main Inductive Hypothesis” (abbreviated by MIH): There is a function w_k which associates $w_k(L) \in A$ to any diagram L with $\text{cr}(L) \leq k$ and the function w_k has the following properties:

III.2.3

$$w_k(U_n) = a_n \text{ where } U_n \text{ is a descending diagram with } n \text{ components} \\ \text{(with respect to some choice of base points)}.$$

III.2.4

$$w_k(L_+) = w_k(L_-) | w_k(L_0)$$

III.2.5

$$w_k(L_-) = w_k(L_+) \star w_k(L_0)$$

for L_+, L_- and L_0 , being related as usually.

III.2.6

$$w_k(L) = w_k(R(L)), \text{ where } R \text{ is Reidemeister move on } L \text{ such that} \\ \text{cr}(R(L)) \text{ is at most } k.$$

Then, as the reader may expect, we want to make the Main Inductive Step, abbreviated as MIS, to obtain the existence of a function w_{k+1} with analogous properties defined on diagrams with at most $k+1$ crossings. Before dealing with the task of making the MIS let us explain that it will really end the proof of the theorem. It is clear that the function w_k satisfying MIS is uniquely determined by properties III.2.3, III.2.4, III.2.5, and the fact that for every diagram there exists a resolving tree with descending leaf diagrams. Thus the compatibility of the functions w_k is obvious and they define a function w defined on diagrams.

The function w satisfies conditions (2) and (3) of Theorem III.1.2, because the functions w_k satisfy such conditions.

If R is a Reidemeister move on a diagram L , then $\text{cr}(R(L))$ equals at most $k = \text{cr}(L) + 2$, whence $w_R(L) = w_k(R(L))$, $w_L = w_k(L)$ and by the properties of w_k we have $w_k(L) = w_k(R(L))$, which implies $w_{R(L)} = w_L$.

It follows that w is an invariant of equivalence classes of oriented diagrams and therefore also of the isotopy class of oriented links.

Now it is clear that w has the required property (1) of Theorem III.1.2, since there exists a descending diagram U_n in the same ambient isotopy class as T_n (e.g. a link diagram without a crossing) and we have $w_{U_n} = a_n$.

The rest of the section §2 will be devoted to the proof of MIS. For a given diagram D with $\text{cr}(D) \leq k+1$ we will denote by \mathcal{D} the set of all diagrams which are obtained from D by a finite number of operations of the kind $\times \rightarrow$ or $\times \rightarrow \succ$.

Of course, once base points $b = (b_1, \dots, b_n)$ are chosen on D , then the same points can be chosen as base points for any $L \in \mathcal{D}$ provided that L is obtained from D by operations of the first type only.

Let us define a function w_b for a given D and b by assigning an element of A to each $L \in \mathcal{D}$. If $\text{cr}(L) < k+1$, then we define

III.2.7

$$w_b(L) = w_k(L).$$

If U_n is a descending diagram with respect to b we put

III.2.8

$$w_b(U_n) = a_n \quad (n \text{ denotes the number of components}).$$

Now we proceed by induction with respect to the number $b(L)$ of bad crossings in L (in the symbol $b(L)$ the letter b works simultaneously for

“bad” and for the choice $b = (b_1, \dots, b_n)$. For a different choice of base points $b' = (b'_1, \dots, b'_n)$ we will write $b'(L)$. Assume that w_b is defined for all $L \in \mathcal{D}$ such that $b(L) < t$ ($t > 0$). Then for the diagram L , $b(L) = t$, let p be the first bad crossing of L (starting from b_1 and proceeding along the diagram). Depending on the crossing p being positive or negative we have $L = L_+^p$ or $L = L_-^p$. We define

III.2.9

$$w_b(L) = \begin{cases} w_b(L_-^p) | w_b(L_0^p) & \text{if } \text{sgn}(p) = + \\ w_b(L_+^p) \star w_b(LJ_0^p) & \text{if } \text{sgn}(p) = -. \end{cases}$$

We will show that w_b is in fact independent of the choice of b and that it has the properties required of w_{k+1} .

III.2.1 Conway Relations for w_b

Let us begin with the proof that w_b has Properties 2.4 and 2.5.

The considered crossing point will be denoted by p . We will restrict our attention to the case $b(L_+^p) > b(L_-^p)$. The opposite case is quite analogous.

We proceed by induction on $b(L_-^p)$. If $b(L_-^p) = 0$, then $b(L_-^p) = 1$ and p is the only one (hence the first) bad crossing of L_+^p . Hence by definition III.2.9 we get

$$w_b(L_+^p) = w_b(L_-^p) | w_b(L_0^p)$$

and further on using the properties of $C6$ we obtain

$$w_b(L_-^p) = w_b(L_+^p) \star w_b(L_0^p).$$

Let us assume now that every diagram L such that $w_b(L_-^p) < t$, with $t \geq 1$, satisfies formulae III.2.4 and III.2.5 for w_b . Let us consider the case $b(L_-^p) = t$.

By the assumption we have $w_b(L_+^p) \geq 2$. Let q be the first bad crossing of the diagram L_+^p . If $q = p$, then by III.2.9 we have

$$w_b(L_+^p) = w_b(L_-^p) | w_b(L_0^p).$$

Let us now consider the case when $q \neq p$. Let $\text{sgn } q = +$, for example. Then by III.2.9 we have

$$w_b(L_+^p) = w_b(L_+^{p \ q}) = w_b(L_+^{p \ q} | w_b(L_+^{p \ q})).$$

Because $w_b(L_{+ -}^{p \ q}) < t$ and $\text{cr} L_{+ 0}^{p \ q} \leq k$, therefore by the inductive hypothesis and by MIH we have

$$w_b(L_{+ -}^{p \ q}) = w_b(L_{- -}^{p \ q}) | w_b(L_{0 -}^{p \ q})$$

and

$$w_b(L_{+ 0}^{p \ q}) = w_b(L_{- 0}^{p \ q}) | w_b(L_{0 0}^{p \ q})$$

hence

$$w_b(L_{+}^p) = (w_b(L_{- -}^{p \ q}) | w_b(L_{0 -}^{p \ q})) | (w_b(L_{- 0}^{p \ q}) | w_b(L_{0 0}^{p \ q})).$$

By the transposition property C3 we obtain

III.2.10

$$w_b(L_{+}^p) = (w_b(L_{- -}^{p \ q}) | w_b(L_{- 0}^{p \ q})) | (w_b(L_{0 -}^{p \ q}) | w_b(L_{0 0}^{p \ q})).$$

On the other hand $b(L_{- -}^{p \ q}) < t$ and $\text{cr}(L_0^p) \leq k$, so using once more the inductive hypothesis and MIH we obtain

III.2.11

$$w_b(L_{-}^p) = w_b(L_{- +}^{p \ q}) = w_b(L_{- -}^{p \ q}) | w_b(L_{- 0}^{p \ q}) w_b(L_0^p) = w_b(L_{0 +}^{p \ q}) = w_b(L_{0 -}^{p \ q}) | w_b(L_{0 0}^{p \ q})$$

Putting together III.2.11 and III.2.10 we obtain

$$w_b(L_{+}^p) = w_b(L_{-}^p) | w_b(L_0^p)$$

as required. If $\text{sgn} q = -$ we should use C4 instead of C3. This completes the proof of Conway Relations for w_b .

III.2.2 Changing Base Points

We will show now that w_b does not depend on the choice of b , provided the order of components of the diagram is not changed. It amounts to the verification that we can replace b_i by b'_i , taken from the same component in such a way that b_i lies after b'_i and there is exactly one crossing point (say p) between them. Let $b' = (b_1, \dots, b'_i, \dots, b_n)$. We want to show that $w_b(L) = w_{b'}(L)$ for every diagram with $k+1$ crossings belonging to \mathcal{D} . Now we will consider the case $\text{sgn} p = +$; the case $\text{sgn} p = -$ is quite analogous.

We use induction with respect to $B(L) := \max(b(L), b'(L))$. We need to consider three cases.

CBP 1 Let us assume that $B(L) = 0$. Then L is a descending diagram with respect to both choices of base points: b just like b' . By III.2.8 we have

$$w_b(L) = a_n = w_{b'}(L).$$

CBP 2 Let us assume that $B(L) = 1$ and $b(L) \neq b'(L)$.

This is possible only when p is a self-crossing point of the i -th component of L . There are two subcases to be considered.

CBP 2 (a) : $b(L) = 1$ and $b'(L) = 0$.

Then L is a descending diagram with respect to b' and by III.2.8 we have

$$w_{b'}(L) = a_n$$

thus by III.2.9 we obtain

$$w_b(L) = w_b(L_+^p) = w_b(L_-^p) | w_b(L_0^p)$$

(we have restricted our attention to the case $\text{sgn} p = +$).

Now $w_b(L_-^p) = a_n$, because $b(L_-^p) = 0$, moreover L_0^p is a descending diagram with respect to a proper choice of base points. Since L_0^p has $n + 1$ components, so $w_b(L_0^p) = a_{n+1}$.

It follows that $w_b(L) = a_n | a_{n+1}$, and since by condition C1 we have $a_n | a_{n+1} = a_n$, so $w_b(L) = a_n = w_{b'}(L)$.

CBP 2 (b) : $B(L) = 0$ and $b'(L) = 1$.

We deal with this case just like with CBP 2(a).

CBP 3 $B(L) = t > 1$ or $B(L) = 1 = b(L) = b'(L)$.

We assume by induction that $w_b(K) = w_{b'}(K)$ for $B(K) < B(L)$. Let q be a bad crossing with respect to b and b' as well. This time we will consider the case $\text{sgn} q = -$. The case $\text{sgn} q = +$ is analogous.

Using the already proven Conway relations for w_b and $w_{b'}$ we obtain:

$$w_b(L) = w_b(L_-^q) = w_b(L_+^q) \star w_b(L_0^q)$$

$$w_{b'}(L) = w_{b'}(L_-^q) = w_{b'}(L_+^q) \star w_{b'}(L_0^q)$$

Since $B(L_+^q) < B(L)$ and $\text{cr}(L_0^q) \leq k$, so by the (local) inductive hypothesis and by MIH we get

$$w_b(L_+^q) = w_{b'}(L_+^q)$$

$$w_b(L_0^q) = w_{b'}(L_0^q)$$

Which imply $w_b(L) = w_{b'}(L)$. This completes the CBP proof.

Since w_b has turned out to be independent of base point changes which preserve the order of components of the diagram, so now we can consider a new function w^0 which associates an element of A to any diagram L with $\text{cr}(L) \leq k + 1$ and with a fixed order of components.

III.2.3 Independence of w^0 of Reidemeister Moves (IRM)

When L is a diagram with a fixed order of components and R is a Reidemeister move on L , then $R(L)$ has a natural order of components yielded by the order of components of L . Assuming that $\text{cr}(R(L)) \leq \text{cr}(L) \leq k + 1$, we will show that $w^0(L) = w^0(R(L))$.

We use induction with respect to the number of bad crossings $b(L)$ for a proper choice of base points $b = (b_1, \dots, b_n)$. This choice must be compatible with the given order of components. Let us choose the base points which lie outside of the part of the diagram involved in the considered Reidemeister move R , so that the same points could work for the diagram $R(L)$ as well.

We need to consider three standard types of Reidemeister moves (Fig.2.1).

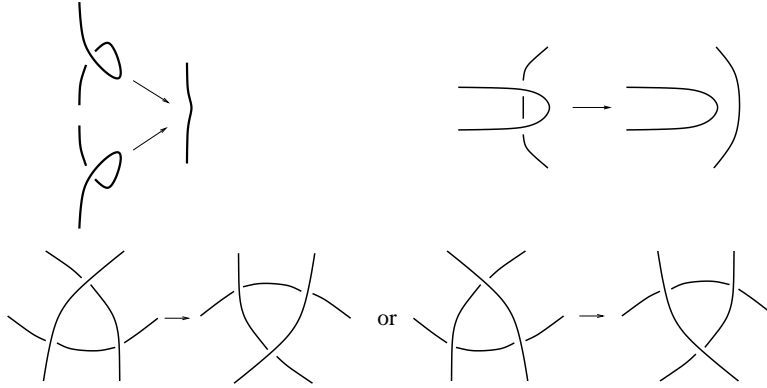


Fig. 2.1

Let us assume that $b(L) = 0$. It is clear then that also $b(R(L)) = 0$ and the number of components remains the same. Thus by III.2.8

$$w^0(L) = w^0(R(L)).$$

Now we assume by induction that $w^0(L) = w^0(R(L))$ when $b(L) < t$. Let us consider the case $b(L) = t$. Assume that there is a bad crossing

p in L which is different from all the crossings involved in the considered Reidemeister move. Assume, for example, that $\text{sgn} p = +$. Then, by the inductive hypothesis, we have

III.2.12

$$w_0(L_-^p) = w^0(R(L_-^p)).$$

and by MIH we obtain

III.2.13

$$w^0(L_0^p) = w^0(R(L_0^p)).$$

Now by the Conway relation III.2.4, which has been already verified for w^0 we have

$$w^0(L) = w^0(L_+^p) = w^0(L_-^p) | w^0(L_0^p)$$

$$w^0(R(L)) = w^0(R(L)_+^p) = w^0(R(L)_-^p) | w^0(R(L)_0^p).$$

Since $R(L_-^p) = R(L)_-^p$ and obviously $R(L_0^p) = R(L)_0^p$, so by III.2.12 and III.2.13 we will get

$$w^0(L) = w^0(R(L)).$$

It remains to consider the case, when all the bad crossings of L , lie in the part of diagram involved in the considered Reidemeister move. Let us consider each of the three types of Reidemeister moves separately. The most complicated case is that of a Reidemeister move of the third type. Before we deal with it, let us formulate the following observation:

Whatever the choice of base points, the crossing point of the top arc and the bottom arc (e.g. q in Fig. 2.2) cannot be the only bad point of the diagram.

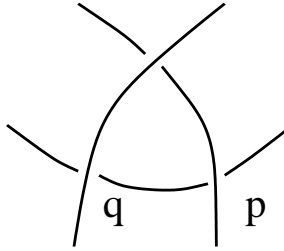


Fig. 2.2

The proof of the above observation amounts to an easy case by case checking. This observation makes the following induction possible. We can assume that we have a bad crossing between the middle arc and either the lower or the upper arc. Let us consider, for example, the situation described by Fig. 2.2. Subsequently we need to consider two subcases, according to $\text{sgnp} = +$ or $-$.

Let us assume that $\text{sgnp} = -$. Then by Conway relations:

$$w^0(L) = w^0(L_-^p) = w^0(L_+^p) \star w^0(L_0^p)$$

$$w^0(R(L)) = w^0(R(L)_-^p) = w^0(R(L)_+^p) \star w^0(R(L)_0^p).$$

By the inductive hypothesis and by the equation $R(L)_+^p = R(L_+^p)$ we obtain:

$$w^0(L_+^p) = w^0(R(L)_+^p).$$

Since $R(L)_0^p$ is obtained from L_0^p by two Reidemeister moves of the second type (Fig. 2.3), thus by MIH $w^0(R(L)_0^p) = w^0(L_0^p)$ and then follows the equality

$$w^0(L) = w^0(R(L)).$$

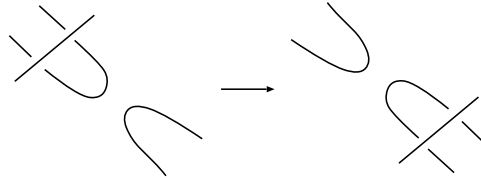


Fig. 2.3

Let us assume now that $\text{sgnp} = +$. Then by Conway relations

$$w^0(L) = w^0(L_+^p) = w^0(L_-^p) | w^0(L_0^p)$$

$$w^0(R(L)) = w^0(R(L)_+^p) = w^0(R(L)_-^p) | w^0(R(L)_0^p).$$

By the inductive hypothesis and equality $R(L)_-^p = R(L_-^p)$ we obtain:

$$w^0(L_-^p) = w^0(R(L)_-^p).$$

Now, L_0^p and $R(L)_0^p$ are essentially the same diagrams (Fig. 2.4), thus $w^0(L_0^p) = w^0(R(L)_0^p)$ and in the end we obtain an equality.

$$w^0(L) = w^0(R(L)).$$

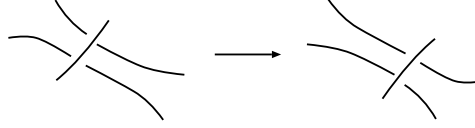


Fig. 2.4

Reidemeister moves of the first type.

The base points can always be chosen in such a way that the crossing point involved in the move is good. Thus $b(L) = b(R(L)) = 0$ and $w^0(L) = w^0(R(L))$.

Reidemeister moves of the second type.

There is only one case when we cannot choose base points to secure that the points involved in the move are good. It happens when the arcs involved are parts of different components and the lower arc is a part of the smaller component in the ordering. In this case both the crossing points are bad and of the opposite signs. Let us consider the case shown on Fig.2.5.

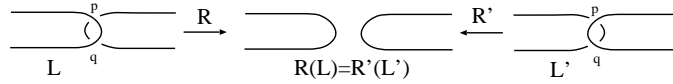


Fig. 2.5

By the inductive hypothesis we have

$$w^0(L') = w^0(R'(L')) = w^0(R(L)).$$

Using the already proven Conway relations, conditions C6 and C7, and MIH, if necessary, it can be proved that $w^0(L) = w^0(L')$. Let us discuss in detail the case involving MIH. It occurs when $\text{sgn} p = +$. Then we have

$$w^0(L) = w^0(L_-^q) = w^0(L_+^q) \star w^0(L_0^q) = (w^0(L_+^{q\ p_-}) | w^0(L_+^{q\ p_0})) \star w^0(L_0^q).$$

But $L_+^{q\ p_-} = L'$ and by MIH it follows that $w^0(L_+^{q\ p_0}) = w^0(L_0^q)$ (see Fig.2.6; here diagrams $L_+^{q\ p_0}$ and L_0^q are both obtained from K by a first Reidemeister move).

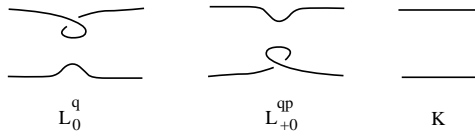


Fig. 2.6

By condition C7 we obtain

$$w^0(L) = w^0(L'),$$

and consequently

$$w^0(L) = w^0(R(L)).$$

The case $\text{sgn } p = -$ is even simpler (see Fig.2.7) and we leave it for the reader.

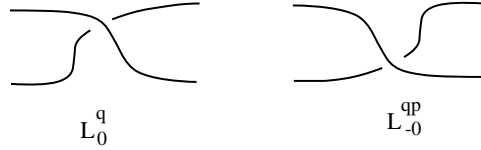


Fig. 2.7 (case $\text{sgn } p = -1$).

This completes the proof of the independence of w^0 of Reidemeister moves.

Remark III.2.14 *If L_i is a trivial component of a diagram L , i.e. L_i has no crossing points, then the specific position of L_i has no effect on $w^0(L)$. Equivalently, If L and L' differ by the position of L_i as in Figure 2.8, then $w^0(L) = w_b(L) = w_b(L') = w^0(L')$. This can be easily achieved by the induction with respect to $b(L)$. Notice however that we cannot use IRM in this case because Reidemeister moves have to increase the number of crossings on the way from L to L' .*

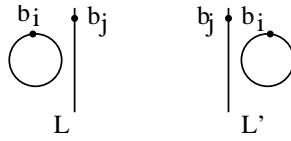


Fig. 2.8

To complete the Main Inductive Step it is enough to prove the independence of w^0 of the order of components. Then we set $w_{k+1} = w^0$. The required properties of w_{k+1} have been already checked.

III.2.4 Independence of the Order of Components (IOC)

It is enough to verify that for a given diagram L , $\text{cr}(L) \leq k + 1$, and for a fixed base points $b = (b_1, \dots, b_i, b_{i+1}, \dots, b_n)$ we have

$$w_b(L) = w_{b'}(L)$$

where $b' = (b_1, \dots, b_{i+1}, b_i, \dots, b_n)$. By induction on $b(L)$ we can easily reduce it to the case of a descending diagram, $b(L) = 0$. To deal with this case we will choose appropriate base points.

The proof will be concluded if we show, that our diagram can be transformed into another one with a smaller number of crossings by a series of Reidemeister moves not increasing the number of crossings. To do it we can use IRM and MIH. This property of a descending diagram is guaranteed by the following lemma.

Lemma III.2.15 *Let L be a diagram with k crossings and a given ordering of components L_1, L_2, \dots, L_n . Then either L has a trivial circle as a component or there is a choice of base points $b = (b_1, \dots, b_n)$, $b_i \in L_i$, such that a descending diagram L^d associated with L and b (that is all the bad crossings of L are changed to good ones) can be changed into a diagram with less than k crossings by a sequence of Reidemeister moves not increasing the number of crossings.*

Proof of lemma III.2.15.

A closed region cut out of the plain by arcs of L is called an i -gon, if it has i vertices (only crossings can be vertices). See Fig.2.9.

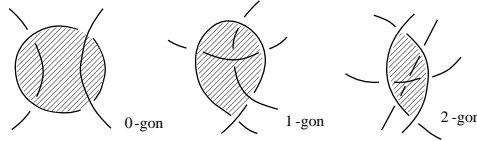


Fig. 2.9

Every i -gon with $i \leq 2$ is called an f -gon (f stands for few). Now let X be an innermost f -gon, that is, an f -gon which does not contain any other f -gon inside.

If X is a 0-gon then we are done because ∂X is a trivial circle. If X is a 1-gon then we are done too, because $\text{int} X \cap L = \emptyset$ so on L^d we can perform a first Reidemeister move decreasing the number of crossings of L^d (Fig.2.10).

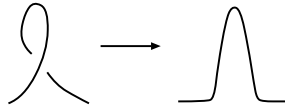


Fig. 2.10

Therefore we assume that X is a 2-gon. Each arc which cuts $\text{int}X$ goes from one edge to another. Furthermore, since no component of L lies fully in X we can choose base points $b = (b_1, \dots, b_n)$ lying outside of X . This has important consequences. If L^d is a descending diagram associated with L and b then each 3-gon in X allows for a Reidemeister move of the third type (i.e. the situation of the Fig.2.11. is impossible).

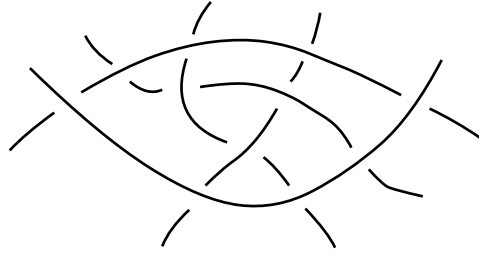


Fig. 2.11

Now we will prove Lemma III.2.15 by induction on the number of crossings of L contained in the 2-gon X . We denote this number by c .

If $c = 2$ then $\text{int}X \cap L = \emptyset$ and we are done thanks to the choice of base points. 2-gon X can be used to make the Reidemeister move of the second type on L^d and to reduce the number of crossings in L^d in this way.

Assume that L has $c > 2$ crossings in X and that Lemma 2.15 has been proved for the number of crossings in X smaller than c .

In order to make the inductive step we need the following lemma.

Lemma III.2.16 *If X is an innermost 2-gon with $\text{int}X \cap L \neq \emptyset$, then there is a 3-gon $\triangle \subset X$ such that $\triangle \cap \partial X \neq \emptyset$ and $\text{int} \triangle \cap L = \emptyset$.*

Before proving Lemma III.2.16 we will show how Lemma III.2.15 follows from it.

Using an 3-gon \triangle from Lemma III.2.16 we can make a Reidemeister move of the third type and reduce the number of crossings L^d in X (compare Fig.2.12).

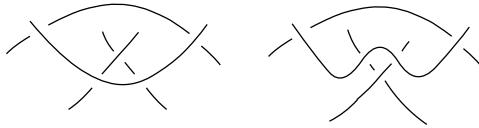


Fig. 2.12

Now either X is an innermost f -gon with less than c crossings in X or X contains an innermost f -gon with less than c crossings. In both cases we can use the inductive hypothesis to complete the proof of the Lemma III.2.15.

Instead of proving Lemma III.2.16 we will show a more general fact of which Lemma III.2.16 is a special case.

Lemma III.2.17 *Let us consider a 3-gon $Y = (a, b, c)$ such that each arc which cuts it goes from the \overline{ab} edge to the \overline{ac} edge with no self-intersections. We allow that Y is a 2-gon considered as a degenerated 3-gon with an edge \overline{bc} collapsed to a point and moreover that there is no f -gon in $\text{int}X$. Furthermore let us assume that there is an arc which cuts $\text{int}X$. Then there is a 3-gon $\Delta \subset Y$ such that $\Delta \cap \overline{ab} \neq \emptyset$ and $\text{int}\Delta$ is not cut by any arc.*

Proof of Lemma III.2.17.

We proceed by induction on the number of arcs in $\text{int}Y \cap L$ (each such an arc cuts \overline{ab} and \overline{ac}). For one arc the Lemma is obvious (Fig.2.13).

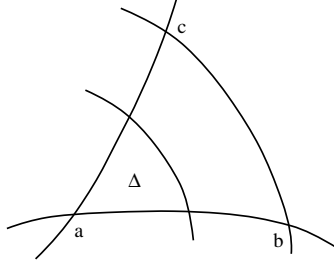


Fig. 2.13

Assume that the Lemma is true for k arcs ($k \geq 1$) and let us consider $(k+1)$ -th arc γ . Let $\Delta_0 = (a_1, b_1, c_1)$ be a 3-gon from the inductive hypothesis with a $\overline{a_1b_1}$ edge included in \overline{ab} (Fig.2.14).

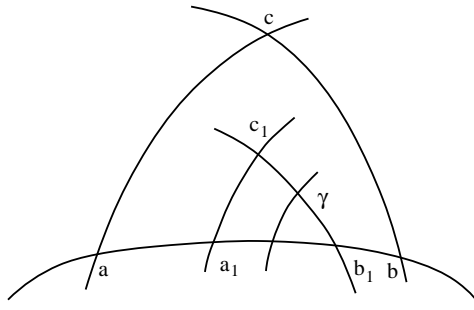


Fig. 2.14

If γ does not cut Δ_o or cuts $\overline{a_1b_1}$, then we are done (Fig.2.14). Therefore let us assume that γ meets $\overline{a_1c_1}$ (at u_1) and $\overline{b_1c_1}$ (at w_1). Let γ cut \overline{ab} at u and \overline{ac} at w (Fig.2.15). Now we should consider two cases:

1. $\overline{uu_1} \cap \text{int}\Delta_0 = \emptyset$ (so $\overline{ww_1} \cap \text{int}\Delta_0 = \emptyset$);

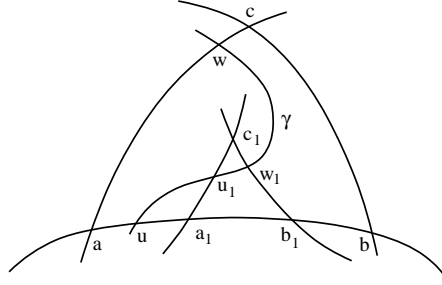


Fig. 2.15

Let us consider the 3-gon ua_1u_1 . No arc can cut the edge $\overline{a_1u_1}$ so each arc which cuts the 3-gon is cut by less than $k+1$ curves. Hence by the inductive hypothesis there is a 3-gon Δ in ua_1u_1 with an edge on $\overline{ua_1}$ and the interior of Δ is cut by no arc at all. In this way Δ is what we need for Lemma III.2.17.

2. $\overline{ww_1} \cap \text{int}\Delta_0 = \emptyset$ (so $\overline{uu_1} \cap \text{int}\Delta_0 = \emptyset$).

This case is analogous to Case 1.

This completes the proof of Lemma III.2.17 — and hence the Lemma III.2.15 as well¹¹.

In this way we have completed the proof of IOC and hence the Main Inductive Step and the proof of Theorem III.1.2.

III.3 Properties of Invariants of Conway Type

We first observe that invariants coming from Conway algebras can be composed, as long as operations used in algebras form an entropic set (Proposition III:1.22). In particular, having a Conway algebra $(A; |)$ we can form a

¹¹A reader familiar with basic fact of the theory of braids may recognize that what we really analyze are braids from \overline{ac} to \overline{ab} . In this language Lemma III:2.17 is intuitively obvious — any braid has the last symbol. It however requires a proof that we actually deal with braids; here “descending” and lack of f -gons should be used as we did in the presented proof.

new Conway algebra $(A; |^n)$. We look more carefully at the example giving a Homflypt polynomial (we consider more familiar skein relation, proposed by H. Morton $v^{-1}P_{L_+} - vP_{L_-} = zP_{L_0}$ with $P_{T_k} = (\frac{v^{-1}-v}{z})^{k-1}$. Thus the Conway algebra yielding this invariant have $A = \mathbb{Z}[v^{\pm 1}, z^{\pm 1}]$, $a_k = (\frac{v^{-1}-v}{z})^{k-1}$, and $a|b = v^2a + vzb$. If we consider composite operation $|^n$ we get

$$\begin{aligned} a|^nb &= (a|^n|b)|b = v^2(a|^n|b) + vzb = v^{2n}a + (v + v^3 + \dots + v^{2n-1})zb = \\ &= v^{2n}a + \frac{v^{2n+1} - v}{v^2 - 1}zb = v^{2n}a + v^n \frac{v^n - v^{-n}}{v - v^{-1}}zb. \end{aligned}$$

The formula holds for any integer n ($|^{-n} = *^n$).

The corresponding Conway skein relation has the form:

$$v^{-n}P_{L_+} - v^nP_{L_-} = \frac{v^n - v^{-n}}{v - v^{-1}}zP_{L_0}.$$

We can offer the following knot theoretical visualization of our composite relation: consider two positive antiparallel crossings as in Figure 3.0. The Conway skein relation yielded by $|$ and applied to L_2 gives $P_{L_2} = v^2P_{L_0} + vP_{L_\infty}$. Now consider the picture with $2n$ antiparallel crossing as in Figure 3.0. Then in order to find $P_{L_{2n}}$ in terms of P_{L_0} and P_{L_∞} we can use once $|^n$ in place of using n times $|$ to get $P_{L_{2n}} = v^{2n}P_{L_0} + v^n \frac{v^n - v^{-n}}{v - v^{-1}}zP_{L_\infty}$.

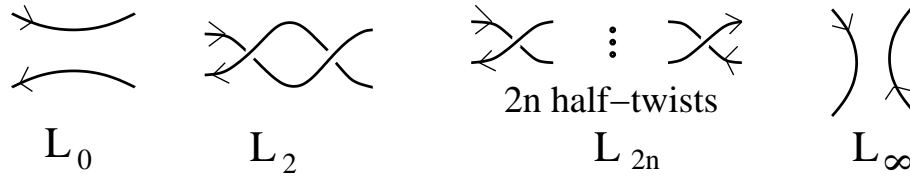


Fig. 3.0; Notation for L_0 , L_2 , L_{2n} , and L_∞ .

Recall that an invariant of links is called an invariant of Conway type if the value of the invariant for L_- and L_0 gives the value for L_+ and the value for L_+ and L_0 gives the value for L_- . We introduce now the relation on oriented links which identifies links that can not be distinguished by any invariant of Conway type. This relation is called Conway skein equivalence and it is denoted by \sim_c [Co-1].

Definition III.3.1 \sim_c is the smallest equivalence relation on ambient isotopy classes of oriented links which satisfies the following condition: Let L'_1

(respectively L'_2) be a diagram of a link L_1 (resp. L_2) with a given crossing p_1 (resp. p_2) such that $\text{sgn } p_1 = \text{sgn } p_2$ and

$$(L'_1)_-^{p_1} \text{sgn } p_1 \sim_c (L'_2)_-^{p_2} \text{sgn } p_2 \quad \text{and} \quad (L'_1)_0^{p_1} \sim_c (L'_2)_0^{p_2}$$

then $L_1 \sim_c L_2$.

From the above definition it follows immediately:

Lemma III.3.2 *Two oriented links are not (Conway) skein equivalent if there exists a Conway type invariant which distinguishes them. In particular, assigning the equivalence class of the Conway relation to a given link is an invariant of Conway type.*

Conway relation can be also described as a “limit” of the sequence of relations. Namely:

(\sim_0) $L_1 \sim_0 L_2$ if L_1 is ambient isotopic to L_2 .

\vdots

\sim_i is the smallest equivalence relation on ambient isotopy classes of oriented links which satisfies the condition:

let L'_1 (resp. L'_2) be a diagram of a link L_1 (resp. L_2) with a crossing p_1 (resp. p_2) such that $\text{sgn } p_1 = \text{sgn } p_2$ and $(L'_1)_-^{p_1} \text{sgn } p_1 \sim_{i-1} (L'_2)_-^{p_2} \text{sgn } p_2$ and $(L'_1)_0^{p_1} \sim_{i-1} (L'_2)_0^{p_2}$ then $L_1 \sim_i L_2$.

Exercise III.3.3 *Show that the smallest relation which contains all the \sim_i relations is Conway equivalence relation.*

We can modify the \sim_i relation without assuming each time that it is an equivalence relation. Namely, by introducing relations $\approx_0, \approx_1, \dots, \approx_i, \dots, \approx_\infty$ in the following way:

$\approx_0 = \sim_0$

\vdots

(\approx_i) $L_1 \approx_i L_2$ if there exist diagrams L'_1 for L_1 and L'_2 for L_2 with crossings p_1 and p_2 respectively such that $\text{sgn } p_1 = \text{sgn } p_2$ and $(L'_1)_-^{p_1} \text{sgn } p_1 \approx_{i-1} (L'_2)_-^{p_2} \text{sgn } p_2$ and $(L'_1)_0^{p_1} \approx_{i-1} (L'_2)_0^{p_2}$

\vdots

\approx_∞ is defined to be the smallest equivalence relation on oriented links which contains all the \approx_i relations.

Problem III.3.4 (1) Are there links which are \sim_c equivalent but not \approx_∞ equivalent?

(2) Are there links which are \sim_i equivalent but are not \approx_i equivalent for any $i, i > 0$?

Exercise III.3.5 One could try to obtain new invariants of links by resolving crossings in pairs, for example, instead of doing it separately. Namely, one could deal with 5 diagrams connected with two crossings on a diagram of a link:

$$L_{\varepsilon_1 \varepsilon_2}^{p \ q}, L_{-\varepsilon_1 \ -\varepsilon_2}^{p \ q}, L_{0 \ -\varepsilon_2}^{p \ q}, L_{-\varepsilon_1 \ 0}^{p \ q}, L_{0 \ 0}^{p \ q}, (\varepsilon_i = + \text{ or } -)$$

and assume further that the value of an invariant for $L_{\varepsilon_1 \varepsilon_2}^{p \ q}$ may be always found on the basis of the value of an invariant for the remaining four diagrams. Show that using this method we will not improve the invariants of the Conway type.¹²

Hint. We can always add one crossing to the diagram without changing the class of ambient isotopy of a link.

Now let us come back to invariants of the Conway type and to the \sim_c equivalence. We start from examples of links which are not ambient isotopic but which are \sim_c equivalent.

Lemma III.3.6 If $-L$ denotes the link obtained from the L link by changing orientation of each component of L then

$$L \sim_c -L.$$

In particular $P_L(x, y) = P_{-L}(x, y)$ where $P_L(x, y)$ is a Jones-Conway polynomial.

Proof is immediate if one notices that the sign of a crossing is not changed when we change L to $-L$. So by building the resolving tree (the same for

¹²When over 26 years ago I formulated Exercise III.3.5 I was not aware of the fact that slight modification of the idea included there would have led to the invariants of links called Vassiliev's invariants (or Vassiliev-Gusarov's invariants or the invariants of the finite type) [Va-1, Bi-Li, Ba, Piu, P-9].

L and $-L$) we show even more than Lemma III.3.6 says: $L \approx_{\text{cr}(L)-1} -L$ where $\text{cr}(L)$ is a minimal number of crossings of diagrams representing L . We leave to a reader further improvement to Lemma III.3.6 using $\text{cr}(L)$ and the number of “bad” crossings in L (as in the proof of the main theorem).

Example III.3.7 The L_1 and L_2 links from Fig.3.1 are \sim_c -equivalent (one should resolve them at the crossings shown on Fig.3.1 and get $L_1 \approx_1 L_2$). To show that L_1 and L_2 are not isotopic one should consider global linking numbers of all its sublinks. Compare Exercise III.3.46.

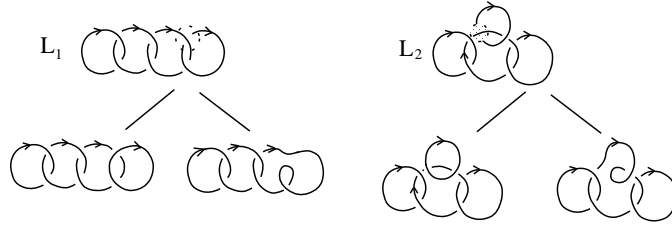


Fig. 3.1

For further examples we need the definition of a tangle and mutation [Co-1]:

Definition III.3.8 (1) A tangle¹³ is a part of a diagram of a link with two inputs and two outputs (Fig.3.2(a)). It depends on an orientation of the diagram which arcs are inputs and which ones are outputs. We distinguish tangles with neighboring inputs and alternated tangles: Fig. 3.2 (b) i (c).

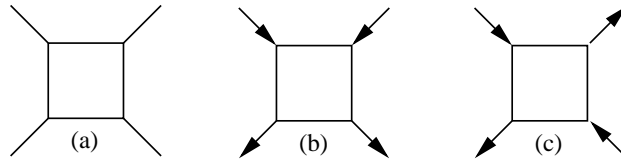


Fig. 3.2

(2) The change of an oriented diagram which concerns just one of its tangles is called a mutation of an oriented diagram which can be of one of the following type (see Fig. 3.2(d)):

¹³We will consider later tangles with n inputs and n outputs, called n -tangles. Then our tangle will be called a 2-tangle.

- (a) m_z is the 180° rotation around the central axis, perpendicular to the plane of the diagram (axis z),
- (b) m_x is the 180° rotation around the horizontal symmetry axis of a square of a tangle (axis x), or
- (c) m_y is the 180° rotation around the vertical symmetry axis of a square of a tangle (axis y).

Moreover, in all the cases, if the need be, we change the orientation of all the components of a tangle into the opposite ones (so that at the change of a tangle the inputs and outputs are preserved). For example a m_z rotation of a tangle (b) requires the change of the orientation of the components of the tangle (compare Fig.3.2(b)).

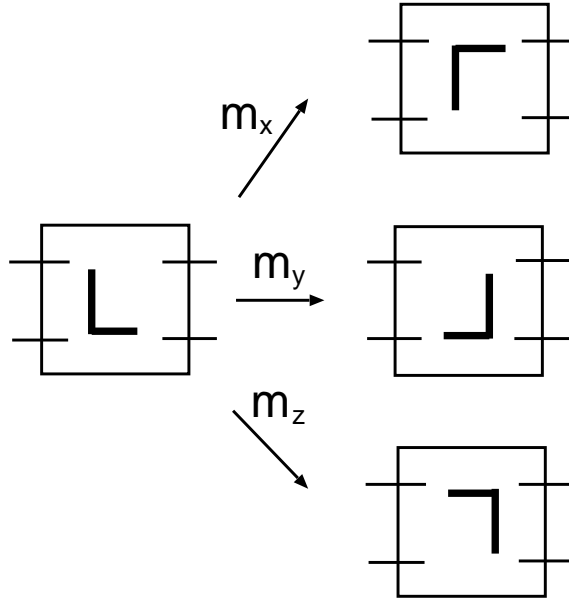


Fig. 3.2(d); three mutations, m_x , m_y , $m_z = m_x m_y$

Lemma III.3.9 ([L-M-1, Hos-1, Gi]) *Two links, L_1 and L_2 , some diagrams of which differ only by a mutation are \sim_c equivalent. More precisely: if cr is the number of crossings in a square of a mutated tangle, then $L_1 \approx_{cr-1} L_2$.*

Proof.

If $cr \leq 1$ then the tangle is (ambient isotopic to) one of the four of Fig.3.3 (in the figure of the tangle we omit possible trivial circles).

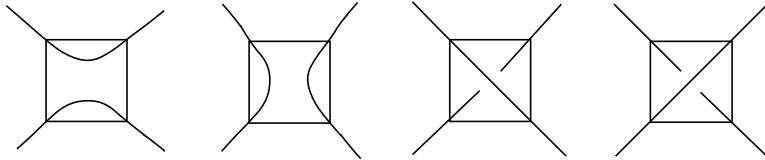


Fig. 3.3

Mutation of a link with respect to the tangles of Fig.3.3 does not change the isotopy class of a link. Further on, to complete the proof, one should use the standard induction with respect to cr and also on the number of bad crossings in the tangle, just like in the proof of the main theorem III.1.2).

The first nontrivial example of a mutation (i.e. the mutation which changes the isotopy class of a knot) was found for the diagrams of 11 crossings.

Example III.3.10 *The Conway knot (Fig.3.4) and the Kinoshita-Teresaka knot (Fig.3.5.) are mutants one of the other (the mutated tangle is marked at Fig.3.4. and 3.5). So these knots are \sim_c -equivalent, even more: they are \approx_1 equivalent. Let us begin resolving the knots from the marked crossings. D. Gabai [Ga-1] has shown that the above knots have different genera (see chapter 3) so they are not isotopic (R. Riley was the first to distinguish these knots [Ri]).*

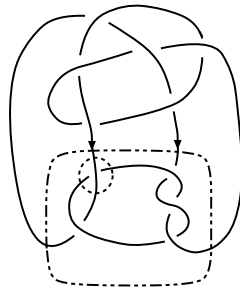


Fig. 3.4. The Conway knot.

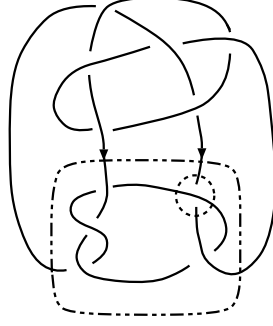


Fig. 3.5. The Kinoshita-Terasaka knot.

Example III.3.11 Let (p_1, p_2, \dots, p_n) be a sequence of n integers. To this sequence we associate an unoriented link $L(p_1, p_2, \dots, p_n)$ which we call pretzel link $L(p_1, p_2, \dots, p_n)$ and which we present on Fig. 3.6.

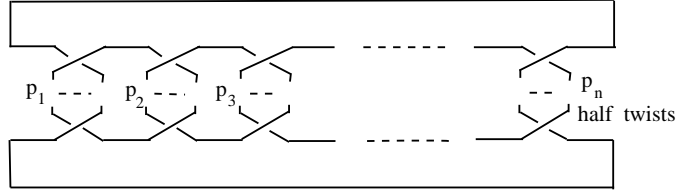


Fig. 3.6

Further, let $L(p_1^{\varepsilon(1)}, p_2^{\varepsilon(2)}, \dots, p_n^{\varepsilon(n)})$ denote the pretzel link $L(p_1, p_2, \dots, p_n)$ oriented in such a way that $\varepsilon(i) = 1$ if all crossings in the i -th column are positive, and $\varepsilon(i) = -1$ if they are negative. To complete the definition of the orientation of the link, we assume that the upper arc of the link is oriented “from the right- to the left-hand-side” (c.f. Fig. 3.7). Clearly, not all possible sequences $p_1^{\varepsilon(1)}, p_2^{\varepsilon(2)}, \dots, p_n^{\varepsilon(n)}$ can be realized by oriented pretzel links. For example, if all p_i and n are odd then $L(p_1, p_2, \dots, p_n)$ is a knot and $\varepsilon(i) = -\text{sgn}(p_i)$. Because of Lemma III.3.9 it follows that for any permutation $\delta_n \in S_n$ we have

$$L(p_1^{\varepsilon(1)}, p_2^{\varepsilon(2)}, \dots, p_n^{\varepsilon(n)}) \sim_c L(p_{\delta(1)}^{\varepsilon(\delta(1))}, p_{\delta(2)}^{\varepsilon(\delta(2))}, \dots, p_{\delta(n)}^{\varepsilon(\delta(n))}).$$

namely, any permutation may be presented by a sequence of transpositions of neighboring numbers and any such a transposition can be realized as a mutation of the pretzel link.

In particular we can go from the pretzel link of two components $L(3, 5, -5^{-1}, -3^{-1}, -3^{-1})$ to its mirror image $L(-3^{-1}, -5^{-1}, -3^{-1}, 5, 3, 3)$ using finite number of mutations. However these links are not ambient isotopic.

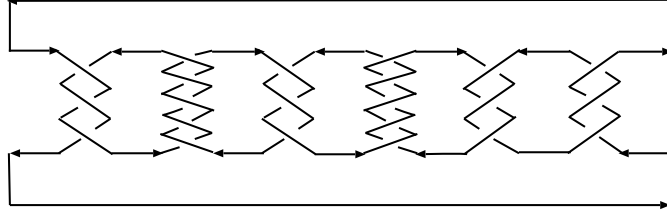


Fig. 3.7. $L(-3^{-1}, -5^{-1}, -3^{-1}, 5, 3, 3)$

Example III.3.12 Consider a diagram of a link D with two alternating tangles inserted as in Fig. 3.9. We use the following convention: $[2n]$ or $2n$ in a square denote tangle as in Fig.3.8.



Fig. 3.8

Moreover, we write \square_{∞} for \square for \square . Let $D_{p,q}$ denote the diagram obtained from D by putting $2p$ into the first tangle and $2q$ into the second. Let us assume moreover that the diagram $D_{\infty,q}$ is equivalent to $D_{p,\infty}$ for every $p \neq q$. Then for $p + q = p' + q'$ we have $D_{p,q} \sim_c D_{p',q'}$. An example which satisfies the above condition was found by T. Kanenobu [Ka-1] (Fig.3.9).

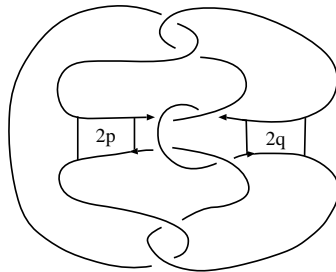


Fig. 3.9

It is easy to notice that in this example $D_{\infty,q}$ and $D_{p,\infty}$ are trivial links of two components. Kanenobu has shown that $D_{p,q}$ is isotopic to $D_{p',q'}$ if and only if $(p, q) = (p', q')$ or (q', p') . That is how we obtain the next family of

examples of nonisotopic links, which are \sim_c -equivalent. Kanenobu proof uses the value of the Jones-Conway polynomial for $D_{p,q}$ (see example III.3.13) and the structure of the Alexander module for $D_{p,q}$ (compare Chapter IV).

Proof of the statement from Example 3.12 can be given by a standard induction on $|p - p'|$ showing $D_{p,q} \approx_{|p-p'|} D_{p',q'}$.

Example III.3.13 Compute the Jones-Conway polynomial for $D_{p,q}$ links of Fig.3.9 and show that $P_{D_{p,q}}(x, y) = P_{D_{p',q'}}(x, y)$ if and only if $p + q = p' + q'$.

Exercise III.3.14 Show that $D_{p,q}$ is isotopic to $D_{q,p}$ for links on Fig.3.9. Hint. Show that Fig.3.10 pictures a link isotopic to $D_{p,q}$.

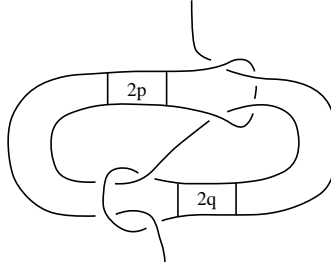


Fig. 3.10

Exercise III.3.15 Show that $D_{p,q}$ link (in Fig. 3.9) is amphicheiral (i.e. isotopic with its mirror image) if and only if $p + q = 0$.

The Kanenobu example may be slightly extended if we allow odd numbers in tangles of the link in Fig. 3.9. [Ka-2].

Let us denote a link the diagram of which is pictured at Fig. 3.9 by $K(m, n)$, where numbers m and n appear in the tangles. So we have $D_{p,q} = K(2p, 2q)$. The orientation of $K(m, n)$ is just implied by the left tangle of Fig.3.9.

Exercise III.3.16 Show that $K(m, n) = -\overline{K}(-m, -n)$.

Exercise III.3.17 Show that $K(m, n) = -\overline{K}(-n, -m) = K(n, m)$.

Exercise III.3.18 Show that $K(m, n) \sim_c K(m', n')$ if and only if $m + n = m' + n'$ and when $m + n$ is even then $m \equiv m' \pmod{2}$.

With the help of a computer M.B. Thistlethwaite has shown that among the 12966 knots with at most 13 crossings there are 30 with the Conway polynomial $1 + 2z^2 + 2z^4$. Examination of these failed to find a pair of knots distinguished by the Homflypt (Jones-Conway) polynomial but not by the Jones polynomial. As a byproduct of these computations Lickorish and Millett [L-M-1] have found the following example.

Example III.3.19 *Let us consider three links pictured on Fig. 3.11 and denoted according to [Ro-1] by 8_8 and 10_{129} , and also 13_{6714} from [This-1].*

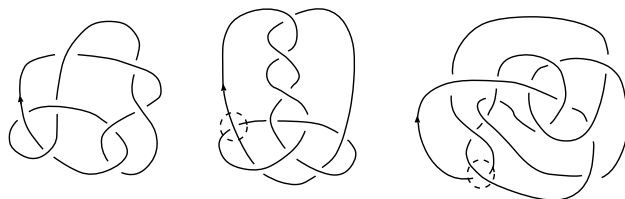


Fig. 3.11

Now changing the encircled crossing of 13_{6714} produces 10_{129} and smoothing that crossing produces T_2 the trivial link of 2 components. Similarly, changing the encircled crossing in 10_{129} gives 8_8 and smoothing it gives T_2 . Hence we have triples $(13_{6714}, 10_{129}, T_2)$ and $(8_8, 10_{129}, T_2)$, both of the form L_+, L_-, L_0 . Therefore $8_8 \sim_c 13_{6714}$. Lickorish and Millett found that 10_{129} and $\bar{8}_8$ (the mirror image of 8_8) have the same Jones-Conway polynomial and they asked if these were \sim_c equivalent (in [L-M-1]). Kanenobu has given the positive answer to this question showing that knots $8_8, 10_{129}$ and 13_{6714} are special cases of his $K(m, n)$ link [Ka-2].

Exercise III.3.20 *Show that $8_8 \approx K(0, -1)$, $10_{129} \approx K(2, -1)$ and $13_{6714} \approx K(2, -3)$.*

Examples which we have described so far have shown limitations of invariants of Conway type. Still, it does not change the fact that for example the Jones-Conway polynomial remains the best single invariant of links. Only the new Kauffman polynomial (discovered in August 1985) may compete with it (compare §5).

For quite some time the question remained open whether the Jones-Conway polynomial is better than the classical Conway polynomial (compare chapter 3 and 4) or the Jones polynomial. M.B. Thistlethwaite searched the tables of knots and found out [L-M-1] that, for example, a knot of 11 crossings

(11₃₈₈ in tables of knots; compare [Per] Fig.3.12), may be differentiated from its mirror image by the Jones-Conway polynomial but not by the Jones polynomial or by Conway polynomial. Because we can compute that:

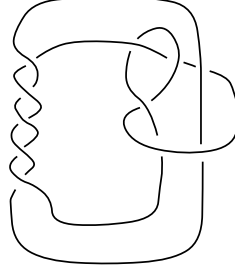


Fig. 3.12. Knot 11₃₈₈.

$$\begin{aligned}
 P_{\overline{11}_{388}}(x, y) &= \\
 5x^{-1}y - 4x^{-1}y^{-1} + 4x^{-2}y^2 - 10x^2 + x^{-2}y^{-2} &+ \\
 + x^{-3}y^3 - 5x^{-3}y + 6x^{-3}y^{-1} + x^{-4} - x^{-4}y^{-2} + 3 & \\
 V_{11_{388}}(t) &= \\
 t^{-2} - t^{-1} + 1 - t + t^2 &= \\
 \nabla_{11_{388}}(z) &= \\
 1 - z^2 - 4z^4 - z^6 &
 \end{aligned}$$

and use the following lemma:

Lemma III.3.21 *If \overline{L} link is a mirror image of link L then their Jones-Conway polynomials satisfy equality:*

$$P_{\overline{L}}(x, y) = P_L(y, x).$$

In particular, for Jones polynomials we have:

$$V_{\overline{L}}(t) = V_L\left(\frac{1}{t}\right)$$

and for Conway polynomials:

$$\nabla_{\overline{L}}(z) = \nabla_L(-z).$$

Proof of the lemma is an easy consequence of the observation that the sign of each crossing changes on the way from a diagram to its mirror image.

Based on the same idea, Lemma III.3.21 can be partially generalized to other invariants yielded by a Conway algebra.

Lemma III.3.22 *Let $\mathcal{A} = \{A, a_1, a_2, \dots, |, \star\}$ be a Conway algebra such that there exists an involution on A (i.e. a mapping, the square of which is identity) $\tau : A \rightarrow A$ satisfying:*

- (i) $\tau(a_i) = a_i$,
- (ii) $\tau(a|b) = \tau(a) \star \tau(b)$.

Then the invariant of links A_L , assigned to this algebra satisfies:

$$A_{\overline{L}} = \tau(A_L).$$

In Examples 1.5 and 1.6 the involution τ is the identity. In Example III.1.8 (defining the Jones-Conway polynomial) the involution is the change of positions of x and y . On the other hand in Example 1.11 $\tau(n, z) = (n, -z)$; the algebra of Example 1.7 has no involution.

It is worthwhile to note that in a Conway algebra build of terms (i.e. “reasonable” words build by the alphabet consisting of a_1, a_2, \dots and $|, \star$) the involution τ exists and it is uniquely determined by the conditions (i) and (ii). It is because τ preserves axioms of the Conway algebra. The Conway algebra build of terms is universal for Conway algebras, that is, there is exactly one homomorphism from it into any other Conway algebra.

Remark III.3.23 *Suppose that \mathcal{A} is a Conway algebra. It may happen that for every pair $u, v \in A$ there exists exactly one element $w \in A$ such that $v|w = u$ and $u \star w = v$. Then we can introduce a new operation $\circ : A \times A \rightarrow A$ putting $u \circ v = w$ (it occurs in Examples 1.6, III.1.7 i III.1.8 but not in examples III.1.5 and III.1.11). Then $a_n = a_{n-1} \circ a_{n-1}$. If operation \circ is well defined then we have an easy formula for invariants of connected and disjoint sums of links. In agreement with a general terminology we call a Conway algebra for which \circ is well defined a Conway quasigroup.*

Definition III.3.24 *A link L is called a splittable link (splittable to L_1 and L_2) if L is a union of two non-empty sublinks L_1 and L_2 , and moreover there exist two disjoint 3-dimensional balls $B_1, B_2 \subset S^3$ such that $L_1 \subset B_1$ and $L_2 \subset B_2$. In such a case we say that L is a disjoint sum of L_1 and L_2 , and we write $L = L_1 \sqcup L_2$.*

Theorem III.3.25 *If $L = L_1 \sqcup L_2$ then*

$$P_{L_1 \sqcup L_2}(x, y) = (x + y)P_{L_1}(x, y)P_{L_2}(x, y)$$

where $P_L(x, y)$ denotes the Jones-Conway polynomial of a link L .

Proof.

There is a diagram L in which L_1 may be separated from L_2 by an ordinary closed curve. We call it a splittable diagram. Now we will prove Theorem III.3.25 for splittable diagrams.

We use the induction with respect to lexicographically ordered pairs $(\text{cr}(L), b(L))$, where $\text{cr}(L)$ denotes the number of crossings in a diagram and $b(L)$ stands for the minimal number of bad crossings over all possible choices of base points.

If $b(L) = 0$ then the theorem III.3.25 holds because L is a trivial link with $n(L)$ components and L_1 and L_2 are trivial links of $n(L_1)$ and $n(L_2)$ components respectively. Thus by the definition

$$\begin{aligned} P_L(x, y) &= \\ (x + y)^{n(L)-1} &= \\ (x + y)(x + y)^{n(L_1)-1}(x + y)^{n(L_2)-1} &= \\ (x + y)P_{L_1}(x, y)P_{L_2}(x, y). \end{aligned}$$

Let us assume that we have already proved Theorem III.3.25 for splittable diagrams satisfying $(\text{cr}(L), b(L)) < (c, b)$. Let p be a bad crossing of diagram L (for example, let $\text{sgn } p = +$). Then for L_-^p and L_0^p the theorem is true by the inductive hypothesis. Let us assume, for example, that $p \in L_1$. Then

$$\begin{aligned} P_L(x, y) &= \\ P_{L_+^p}(x, y) &= \\ \frac{1}{x}(P_{L_0^p}(x, y) - yP_{L_-^p}(x, y)) &= \\ \frac{1}{x}((x + y)P_{(L_1)_0^p}(x, y) \cdot P_{L_2}(x, y) - y(x + y)P_{(L_1)_-^p}(x, y) \cdot P_{L_2}(x, y)) &= \\ (x + y)P_{(L_2)_0^p}(x, y) \cdot (\frac{1}{x}P_{(L_1)_0^p}(x, y) - yP_{(L_1)_-^p}(x, y)) &= \\ (x + y)P_{L_2}(x, y) \cdot P_{L_1}(x, y) \end{aligned}$$

which completes the proof.

In the other cases we proceed similarly.

Definition III.3.26 *An oriented link L is a connected sum of two links L_1 and L_2 (we denote it $L = L_1 \# L_2$) if there exists a sphere $S^2 \subset S^3$ which divides S^3 into two 3-dimensional balls B_1 and B_2 in such a way that S^2 meets L transversally in two points, and if β is an arc in S^2 joining these two points then $(B_1 \cap L) \cup \beta$ (resp. $(B_2 \cap L) \cup \beta$) is isotopic to L_1 (resp. L_2).*

In Chapter IV we analyze a connected sum in detail and in particular we will show that it is uniquely defined for knots (with the sum knots form an abelian semigroup with cancellation property).

Corollary III.3.27 *If $L = L_1 \# L_2$ then*

$$P_L(x, y) = P_{L_1}(x, y) \cdot P_{L_2}(x, y).$$

Proof.

We can find a diagram of L as presented on Fig.3.13. Let us rotate L_2 by 180° twice: once clockwise, next counterclockwise to get two diagrams L_+ and L_- (Fig.3.14). Of course L_+ and L_- are isotopic to L and the diagram L_0 (Fig. 3.14) is the disjoint sum of L_1 and L_2 , so:

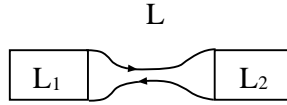


Fig. 3.13

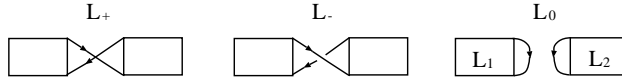


Fig. 3.14

$$xP_L(x, y) + yP_L(x, y) = P_{L_1 \sqcup L_2}(x, y),$$

and therefore

$$(x + y)P_{L_1 \# L_2}(x, y) = P_{L_1 \sqcup L_2}(x, y),$$

This formula and Theorem III.3.25 gives us Corollary III.3.27.

Theorem III.3.25 and Corollary III.3.27 may be generalized to cover the case of A_L invariants yielded by Conway algebras with the operation \circ . First,

let us observe that adding a trivial knot to the given link L changes A_L for $A_L \circ A_L$ (in short A_L^2); Fig.3.15. In particular we obtain a known equality $a_i^2 = a_{i+1}$. Considering Fig.3.14 we get more general formula:

$$A_{L_1 \sqcup L_2} = A_{L_1 \# L_2}^2.$$

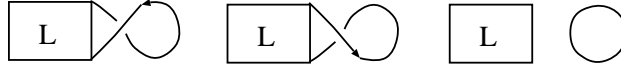


Fig. 3.15

Using a method similar to that of Theorem III.3.25 and Corollary III.3.27 we can prove the following:

Lemma III.3.28 *If a Conway algebra \mathcal{A} admits the operation \circ and for each $w \in A$ there is a homomorphism $\phi_w : A \rightarrow A$ such that*

$$\phi_w(a_1) = w, \phi_w(a_2) = w^2, \phi_w(a_3) = w^4, \dots, \text{etc}$$

then

$$\begin{aligned} A_{L_1 \# L_2} &= \phi_{A_{L_1}}(A_{L_2}) = \phi_{A_{L_2}}(A_{L_1}) \\ A_{L_1 \sqcup L_2} &= (\phi_{A_{L_1}}(A_{L_2}))^2 = (\phi_{A_{L_2}}(A_{L_1}))^2 \end{aligned}$$

Exercise III.3.29 *Show that the algebras of Examples 1.6, III.1.7 and III.1.8 satisfy the assumptions of Lemma III.3.28.*

Problem III.3.30 (i) *Let us consider the equation $a|x = b$ in the universal Conway algebra. Is it possible for this equation to have more than one solution? (the equation $a_1|x = a_2$ may have no solution at all).*

(ii) *Let us assume that for certain diagrams of L and L' and for certain crossings we have $L_+ \sim_c L'_+$ and $L_- \sim_c L'_0$ are true. Is then $L_0 \sim_c L'_0$ true as well?*

Corollary III.3.27 can be generalized in the following way:

Definition III.3.31 (i) *Consider an alternating tangle A . There are two methods of obtaining a link from the tangle A . They are marked $N(A)$ (numerator) and $D(A)$ (denominator) according to Fig.3.16.*

Let A^N denote $P_{N(A)}(x, y)$ and $A^D = P_{D(A)}(x, y)$ (i.e. the respective value of the Jones-Conway polynomial).



Fig. 3.16

(ii) Having two alternating tangles A and B we can define their sum just like on Fig.3.17. Let us notice that $D(A+B) = D(A)\#D(B)$.

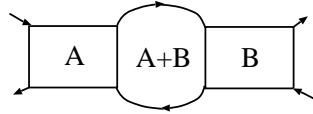


Fig. 3.17

Lickorish and Millett [L-M-1] have generalized Conway's result [Co-1] concerning Conway polynomial by showing:

Theorem III.3.32 1. $(1 - (x + y)^2)(A + B)^N = (A^N B^D + A^D B^N) - (x + y)(A^N B^N + A^D B^D)$
 2. $(A + B)^D = A^D \cdot B^D$

Proof.

Part (ii) has been already proved in Corollary III.3.27.

Part (i) will be proved by induction with respect to ordered pairs $(cr(B), b(B)) =$ (number of crossings in the tangle B , minimal number of bad crossings in B), similarly as in the proof of Theorem III.3.25. For a tangle B we can find a resolving tree leaves of which are the tangles of Fig.3.18, possibly with a certain number of trivial circles.

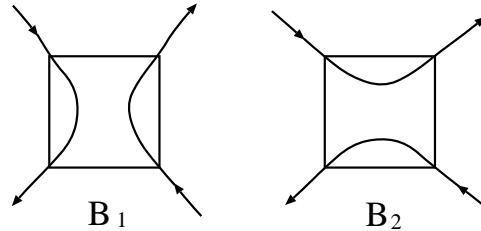


Fig. 3.18

Because the trivial circles appear in $A + B$ as well, they can be omitted in further considerations. As $N(B_1)$ and $D(B_2)$ are trivial knots and both

$D(B_1)$ and $N(B_2)$ are trivial links of two components, and moreover $N(A + B_1) = D(A)$ and $N(A + B_2) = N(A)$ so that $B_1^N = B_2^D = 1$, $B_1^D = B_2^N = (x + y)$, $(A + B_1)^N = A^D$ as well as $(A + B_2)^N = A^N$. It follows that

$$(1 - (x + y)^2)(A + B_1)^N = (1 - (x + y)^2)A^D = (A^N + (x + y) + A^D) - (x + y)(A^N + A^D(x + y))$$

similarly

$$(1 - (x + y)^2)(A + B_2)^N = (1 - (x + y)^2)A^N = (A^N + (x + y) + A^D) - (x + y)((x + y)A^N + A^D).$$

Thus we have proved Theorem III.3.32(i) for $A + B_1$ and $A + B_2$.

Now the immediate verification shows that if the formula holds for B_- and B_0 , then it is true for B_+ as well, and similarly if it holds for B_+ and B_0 , then it is true for B_- . This allows the inductive step and completes the proof of Theorem III.3.32(i).

Exercise III.3.33 Show that the Jones-Conway polynomial distinguishes square knot and granny knot (Fig. 3.19).

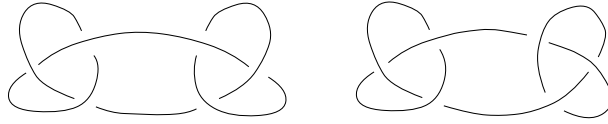


Fig. 3.19

Corollary III.3.34 ([Co-1].) Let us define the fraction of a tangle A as follows:

$$F(A) = \frac{\nabla_{N(A)}(z)}{\nabla_{D(A)}(z)}$$

where $\nabla(z)$ is a Conway polynomial. We do not cancel common factors of numerator and denominator.

Then $F(A + B) = F(A) + F(B)$.

Example III.3.35 Let A be a tangle pictured at Fig. 3.20. Then $F(A) = \frac{z}{1}$.

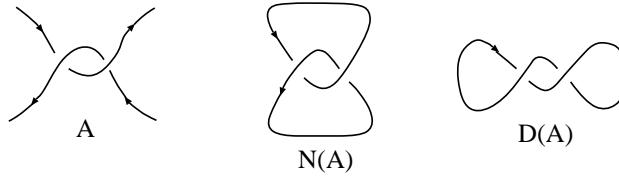


Fig. 3.20

Problem III.3.36 *Let \mathcal{A} be a Conway algebra with the operation \circ and the homomorphism ϕ_w . Find the formula for value of the invariant yielded by the algebra for the numerator of a sum of two tangles.*

In the subsequent chapters we will show how to use theorem III.3.32 for computing the Jones-Conway polynomial of certain classes of links; namely, links with two bridges, pretzel links and Montesinos links.

J. Birman [Bi-2] (and, independently, M. Lozano and H. Morton [Lo-Mo]) has found examples of knots which are not isotopic but have the same Jones-Conway polynomials. In [L-M-1] it was noticed that these links are not \sim_c equivalent for they have different signatures (see §5 and chapter III).

In [P-T-2] it was proved that these links are not algebraically equivalent (i.e. they can not be distinguished by any invariant yielded by a Conway algebra). Fig.3.21. shows the simplest pair of Birman knots.

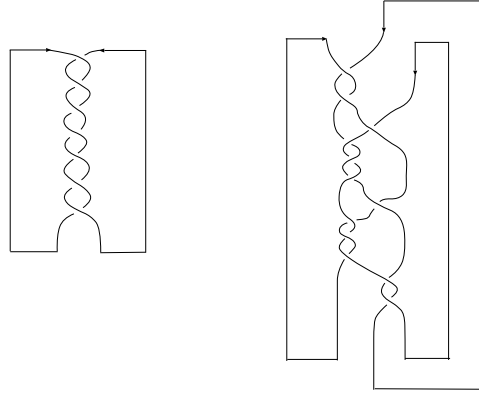


Fig. 3.21

Problem III.3.37 (i) *Do there exist links which have the same Jones-Conway polynomial but are not algebraically equivalent?*

(ii) *Do there exist links which are not algebraically equivalent, still they have the same value of an invariant yielded by any finite Conway algebra?*

Many algebraic properties of the Jones-Conway polynomial are known. Mostly these which relate its special substitutions with old invariants of links [L-M-1, L-M-2, Mur-1, Mo-3] or [F-W]. We will discuss these properties in respective chapters later on. Now we will present two quite elementary properties of Jones-Conway polynomials.

Lemma III.3.38 (i) *If L is a link of odd number of components then all the monomials of $P_L(x, y)$ polynomial are of even degree. If L is a link of an even number of components then these monomials are of an odd degree.*

(ii) *For every link L , $P_L(x, y) - 1$ is divisible by $x + y - 1$, in particular, the Jones-Conway polynomial of a link cannot be identically zero.*

(iii) *$P_L(x, y) + (-1)^{\text{com}(L)}$ is divisible by $x + y + 1$; $\text{com}(L)$ here means the number of components of link L .*

(iv) *$P_L(x, y) - (x + y)^{\text{com}(L)-1}$ is divisible by $(x + y)^2 - 1$.*

Proof: It is easy to check that the conditions (i)-(iv) are true for trivial links. Then, it is enough to establish that if they are true for L_- L_0 (resp. L_+ and L_0) then they are true for L_+ (resp. L_-) as well. \square

In fact one can improve a little Lemma III.3.38 using the same inductive proof. For this one should consider the skein relation in slightly more general form:

$$xP_{L_+} + yP_{L_-} = zP_{L_0}$$

and do not assume that z is invertible. Then the divisibility of $P_L - P_{T_{\text{com}(L)}}$ for knots can be formulated in more general manner than for any link (we did observe this already in the case of the Jones polynomial (see Chapter I). The value of the invariant for a trivial link of n components is $P_{T_n} = (\frac{x+y}{z})^{n-1}$, thus if z is not necessarily invertible we should work with invariants in the ring $A = \mathbb{Z}[x^{\pm 1}, y^{\pm 1}, z, \frac{x+y}{z}]$ or more formally $A = \mathbb{Z}[x^{\pm 1}, y^{\pm 1}, z, d]/(zd - (x + y))$. The idea of working with this ring of invariants is used and developed when we work with periodic link, Chapter VIII. With the above notation we have:

Proposition III.3.39

(i) *For a link L of n components $P_L(x, y, z) - P_{T_n}(x, y, z)$ is divisible by $(x + y)\frac{x+y}{z} - z$.*

(ii) *For the Jones polynomial, that is $x = t^{-1}$, $y = -t$, and $z = \sqrt{t} - \frac{1}{\sqrt{t}}$, we get: $V_L(t) - V_{T_n}(t)$ is divisible by $t^3 - 1$.*

(iii) *For a knot K , $P_K(x, y, z) - 1$ is divisible by $(x + y)^2 - z^2$.*

- (iv) For the Jones polynomial of a knot K , $V_K(t) - 1$ is divisible by $(t - 1)(t^3 - 1)$.

Exercise III.3.40

- (i) Show that for a positive Hopf link, H_+ we have:

$$\frac{P_{H_+} - P_{T_2}}{(x+y)^{\frac{x+y}{z}} - z} = \frac{1}{x}.$$

- (ii) Show that for a negative Hopf link, H_- we have:

$$\frac{P_{H_-} - P_{T_2}}{(x+y)^{\frac{x+y}{z}} - z} = \frac{1}{y}.$$

- (iii) Show that for a positive (right handed) trefoil knot, $\bar{3}_1$ we have:

$$\frac{P_{\bar{3}_1} - 1}{((x+y)^2 - z^2)} = -\frac{1}{x^2}.$$

- (iv) Show that for a negative (left handed) trefoil knot, 3_1 we have:

$$\frac{P_{3_1} - 1}{((x+y)^2 - z^2)} = -\frac{1}{y^2}.$$

- (v) Show that for a figure eight knot, 4_1 we have:

$$\frac{P_{4_1} - 1}{((x+y)^2 - z^2)} = -\frac{1}{xy}.$$

- (v) Let K_n be a twist knot of $n+2$ crossings (e.g. $K_2 = 4_1$ and $K_{-2} = \bar{3}_1$). Show that for $n = 2k$ we have:

$$\frac{P_{K_{2k}} - 1}{((x+y)^2 - z^2)} = -\frac{y^k - (-1)^k x^k}{xy^k(x+y)}.$$

Show that the formula holds also for negative k and find the formula in the case of twist knots with n odd.

In the definition of the Jones-Conway polynomial one could try to replace the equation 1.9 by

III.3.41

$$xw_1 + yw_2 = zw_0 - v.$$

or, in the case z is invertible, by

III.3.42

$$xw_1 + yw_2 = w_0 - z.$$

Indeed, it leads to the invariant of links in $\mathbb{Z}[x^{\pm 1}, y^{\pm 1}, z]$ [P-T-1], however this polynomial does not distinguish links better than the Jones-Conway polynomial (it was noticed in the Spring of 1995 by the referee of [P-T-1], and later, independently by O.Ya.Viro [Vi]).¹⁴ Namely:

Exercise III.3.43 (i) Show that the following algebra is a Conway algebra. $\mathcal{A} = \{A, a_1, a_2, \dots, |, \star\}$ where $A = \mathbb{Z}[x^{\pm 1}, y^{\pm 1}, z]$, $a_1 = 1, a_2 = x + y + z, \dots, a_i = (x + y)^{i-1} + z(x + y)^{i-2} + \dots + z(x + y) + z, \dots$. We define the $|$ operation and \star as follows: $w_2|w_0 = w_1$ and $w_1 \star w_0 = w_2$ where

$$xw_1 + yw_2 = w_0 - z; w_1, w_2, w_0 \in A.$$

(ii) Show that the invariant of links $w_L(x, y, z)$, defined by a Conway algebra of (i) satisfies

$$w_L(x, y, z) = w_L(x, y, 0) + z\left(\frac{w_L(x, y, 0) - 1}{x + y - 1}\right)$$

and that

$$w_L(x, y, 0) = P_L(x, y).$$

Hint for (ii). Notice that

$$a_i = (x + y)^{i-1} + z\left(\frac{(x + y)^{i-1} - 1}{x + y - 1}\right).$$

Definition III.3.44 Every invariant of links can be used to build a better one. This will be called a weighted simplex of the invariant. Namely, if w is an invariant and L is a link of n components L_1, L_2, \dots, L_n , then we consider an $(n - 1)$ -dimensional simplex $\Delta^{n-1} = (q_1, q_2, \dots, q_n)$. To each face $(q_{i_1}, \dots, q_{i_k})$ of a simplex Δ^{n-1} we assign value $w_{L'}$, where $L' = L_{i_1} \cup \dots \cup L_{i_k}$.

¹⁴This is the case as long as we assume that a coefficient of w_0 is invertible.

We say that two weighted simplexes are equivalent if there exists a permutation of their vertices which preserves weights of faces.

Of course, the weighted simplex of an invariant of ambient isotopy classes of oriented links is also an invariant of ambient isotopy classes of oriented links.

Example III.3.45 ([Bi-2].) *The links of Fig.3.22 have the same Jones-Conway polynomial (as well as the signature — see §4 and Chapter IV); still they are easy distinguishable by the weighted simplex of the global linking numbers. (Example III.1.11).*

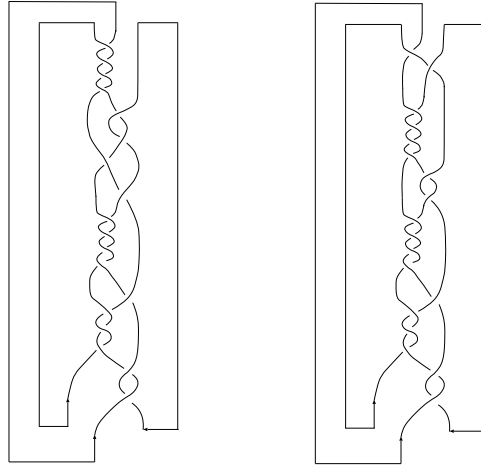


Fig. 3.22

Exercise III.3.46 *Show that the links L_1 and L_2 of example III.3.7 can be distinguished by the weighted simplex of global linking numbers.*

III.4 Partial Conway Algebras.

To get a Conway type link invariant with values in a set A it is not necessary for the operations $|$ and \star to be defined on the whole product $A \times A$. Similarly there is no need for the relations $C3 - C5$ to be satisfied for all quadruples of $A \times A \times A \times A$. It is enough for the operations to be defined and for the relations to be satisfied merely in the case when the geometrical situation requires that. The above observation leads to the definition of geometrically sufficient partial Conway algebras which define the invariants of links of Conway type. These invariants can be more subtle than the ones obtained

by Conway algebras (e.g. signature). The results we present here are based on [P-T-1, P-T-2, P-1].

Definition III.4.1 *A partial Conway algebra is a quadruple (A, B_1, B_2, D) where B_1 and B_2 are subsets of $A \times A$ and D is a subset of $A \times A \times A \times A$, together with 0-argument operations a_1, a_2, \dots and two 2-argument operations $|$ and \star which are defined on B_1 and B_2 , respectively. The operations $|$ and \star are assumed to satisfy equalities C1 – C7 of Definition 1.1, provided both sides of the respective equality are defined, and additionally $(a, b, c, d) \in D$ in the case of equalities C3 – C5.*

Definition III.4.2 *We say that a partial Conway algebra $\mathcal{A} = (A, B_1, B_2, D, a_1, a_2, \dots, |, \star)$ is geometrically sufficient if and only if the following two conditions are satisfied.*

- (i) *For every resolving tree of a link all the operations that are necessary to compute the root value are admissible, that is, all the intermediate values are in the sets where $|$ and \star are defined.*
- (ii) *Let p_1, p_2 be two different crossings of a diagram L . Let us consider the diagrams $L_{\varepsilon_1 \varepsilon_2}^{p_1 p_2}, L_{\varepsilon_1 0}^{p_1 p_2}, L_{0 \varepsilon_2}^{p_1 p_2}, L_{0 0}^{p_1 p_2}$, where $\varepsilon_i = -\text{sgn } p_i$ and let us choose for them the resolving trees $T_{p_1, p_2}, T_{p_1, 0}, T_{0, p_1}, T_{0, 0}$ respectively. Denote the root values of these trees by $w_{p_1, p_2}, w_{p_1, 0}, w_{0, p_2}, w_{0, 0}$ respectively. Now, we assume that in the above case always $(w_{p_1, p_2}, w_{p_1, 0}, w_{0, p_2}, w_{0, 0}) \in D$.*

Condition (ii) means that the resolving trees of L of Fig.4.1 give the same values at the roots of the trees.

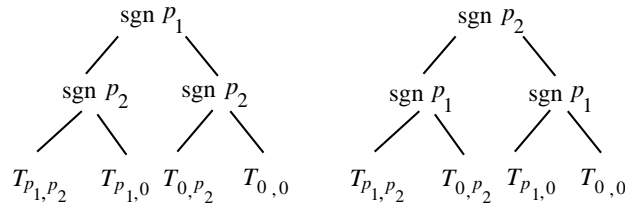


Fig. 4.1

The proof of Theorem III.1.2 can be used, without changes, for the case of a geometrically sufficient partial Conway algebra.

Theorem III.4.3 *Let \mathcal{A} be a geometrically sufficient partial Conway algebra. Then there exists a unique invariant w which associates an element from \mathcal{A} to any skein equivalence class of links and the invariant w satisfies the following conditions:*

1. $w_{T_n} = a_n$
2. $w_{L_+} = w_{L_-} | w_{L_0}$
3. $w_{L_-} = w_{L_+} \star w_{L_0}$.

Similarly as in the case of Conway algebras, the conditions $C1 - C7$ for partial Conway algebras are not independent. Namely, we have the following result, the proof of which is left to the reader (but see the comment after the lemma).

Lemma III.4.4 *Let $(A, B_1, B_2, a_1, a_2, \dots, |, \star)$ be a partial algebra such that:*

1. *The property (i) in Definition III.4.2 is satisfied.*
2. *The property (ii) in Definition III.4.2 is satisfied for each pair of crossings of positive sign, i.e. the resolving trees of L from Fig. 4.1 give the same values at their roots if $\text{sgn } p_1 = \text{sgn } p_2 = +$.*
3. *The conditions $C1$, $C6$ and $C7$ are satisfied if both sides of the equations are defined.*

If D is a subset of $A \times A \times A \times A$ for which the condition $C3$ is satisfied then $\mathcal{A} = (A, B_1, B_2, D, a_1, a_2, \dots, |, \star)$ is a geometrically sufficient partial Conway algebra.

In our proof, we follow that of Theorem III:1.3 but we should be sure that we are moving inside the domain of our partial algebra. Thus assume (3), (6) and (7) and we will prove (4). Consider a link diagram L and its two crossing of different signs, p_1 and p_2 . We can assume that $L = L_{+}^{p_1 p_2} -$. Then when resolving first p_1 and then p_2 we have allowed expression:

$$(w_{L_{-}^{p_1 p_2} +} * w_{L_{-}^{p_1 p_2} 0}) | (w_{L_0^{p_1 p_2} +} * w_{L_0^{p_1 p_2} 0})$$

If we resolve first p_2 and then p_1 we get another allowed expression

$$(w_{L_{-}^{p_1 p_2} +} | w_{L_0^{p_1 p_2} +}) * (w_{L_{-}^{p_1 p_2} 0} | w_{L_0^{p_1 p_2} 0})$$

and our goal is to show that they are equal. Now $(w_{L_-^{p_1 p_2}} * w_{L_-^{p_1 p_2}})$ is from the definition equal to $w_{L_-^{p_1 p_2}}$ and $w_{L_-^{p_1 p_2}} | w_{L_0^{p_1 p_2}}$ is equal to $w_{L_-^{p_1 p_2}}$.

Now we will construct two examples of geometrically sufficient partial Conway algebras and we will discuss the invariant of links which are defined by these two algebras. Let us begin with an example which leads to a direct generalization of the Jones-Conway polynomial. Instead of equations III.1.9 or III.3.42 we use other equations depending on the number of components of L_+ , L_- and L_0 .

Example III.4.5 *The following partial algebra \mathcal{A} is a geometrically sufficient partial Conway algebra:*

$$\begin{aligned}
A &= N \times \mathbb{Z} [y_1^{\pm 1}, x_2'^{\pm 1}, z_2'^{\pm 1}, x_1^{\pm 1}, z_1, x_2^{\pm 1}, z_2, x_3^{\pm 1}, z_3, \dots] \\
B_1 = B_2 &= \{((n_1, w_1), (n_2, w_2)) \in A \times A : |n_1 - n_2| = 1\}, D = A \times A \times A \times A \\
a_1 &= (1, 1) \\
a_2 &= (2, x_1 + y_1 + z_1) \\
&\vdots \\
a_n &= (n, \prod_{i=1}^{n-1} (x_i + y_i) + z_1 \cdot \prod_{i=2}^{n-1} (x_i + y_i) + \dots + z_{n-2} (x_{n-1} + y_{n-1}) + z_{n-1}) \\
&\vdots
\end{aligned}$$

where $y_i = x_i \cdot \frac{y_1}{x_1}$. To define operations $|$ and \star we consider the following system of equations:

$$\begin{aligned}
(1) \quad & x_1 w_1 + y_1 w_2 = w_0 - z_1 \\
(2) \quad & x_2 w_1 + y_2 w_2 = w_0 - z_2 \\
(2') \quad & x_2' w_1 + y_2' w_2 = w_0 - z_2' \\
(3) \quad & x_3 w_1 + y_3 w_2 = w_0 - z_3 \\
(3') \quad & x_3' w_1 + y_3' w_2 = w_0 - z_3' \\
&\vdots \\
(i) \quad & x_i w_1 + y_i w_2 = w_0 - z_i \\
(i') \quad & x_i' w_1 + y_i' w_2 = w_0 - z_i'
\end{aligned}$$

where $y'_i = \frac{x'_i y_1}{x_1}$, $x'_i = \frac{x'_2 x_1}{x_{i-1}}$ and z'_i is defined inductively so that it satisfies the equality

$$\frac{z'_{i+1} - z_{i-1}}{x_1 x'_2} = (1 + \frac{y_1}{x_1}) (\frac{z'_i}{x'_i} - \frac{z_i}{x_i}).$$

Now we define $(n, w) = (n_1, w_1) | (n_2, w_2)$ and, respectively, $(n, w) = (n_1, w_1) \star (n_2, w_2)$ in the following way: we set $n := n_1$ and further

1. if $n_1 = n_2 - 1$ then we use equation (n) ($n_1 = n$) to determine w , namely $x_n w + y_n w_1 = w_2 - z_n$ (respectively, $x_n w_1 + y_n w = w_2 - z_n$),
2. if $n_1 = n_2 + 1$ then we use equation (n') to determine w , namely $x'_n w + y'_n w_1 = w_2 - z'_n$ (respectively, $x'_n w_1 + y'_n w = w_2 - z'_n$).

We shall prove that such \mathcal{A} is a geometrically sufficient partial Conway algebra.

It is easy to check that first coordinates of elements from A satisfy relations $C1 - C7$ (they define the number of components in the link c.f. Example III.1.5. Also, it is not hard to check that A satisfies relations $C1$, $C2$, $C6$ and $C7$. Therefore we concentrate on relations $C3 - C5$.

It is convenient to use the following notation: if $w \in A$ then $w = (|w|, F_w)$ and for

$$w_1 | w_2 = (|w_1|, F_{w_1}) | (|w_2|, F_{w_2}) = (|w|, F_w) = w$$

we write

$$F_w = \begin{cases} F_{w_1} |_n F_{w_2} & \text{if } n = |w_1| = |w_2| - 1 \\ F_{w_1} |_{n'} F_{w_2} & \text{if } n = |w_1| = |w_2| + 1. \end{cases}$$

We also use a similar notation for the operation \star .

In order to verify $C3 - C5$ we have to consider three main cases:

1. $|a| = |c| - 1 = |b| + 1 = n$; $a, b, c, d \in A$.

Both sides of relations $C1 - C5$ are defined if and only if $|d| = n$. The relation $C3$ is then reduced to the following equation:

$$(F_a |_{n'} F_b) |_n (F_c |_{n'} F_d) = (F_a |_n F_b) |_{n'} (F_c |_n F_d).$$

From this we get:

$$\frac{1}{x_n x'_{n+1}} \cdot F_d - \frac{y'_{n+1}}{x_n x'_{n+1}} \cdot F_c - \frac{y_n}{x_n x'_n} \cdot F_b + \frac{y_n y'_n}{x_n x'_n} \cdot F_a -$$

$$\begin{aligned}
& \frac{z'_{n+1}}{x_n x'_{n+1}} - \frac{z_n}{x_n} + \frac{y_n z'_n}{x_n x'_n} = \\
& \frac{1}{x'_n x_{n-1}} \cdot F_d - \frac{y'_{n-1}}{x'_n x_{n-1}} \cdot F_b - \frac{y'_n}{x_n x'_n} \cdot F_c + \frac{y_n y'_n}{x_n x'_n} \cdot F_a - \\
& \frac{z_{n-1}}{x'_n x_{n-1}} - \frac{z'_n}{x'_n} + \frac{y'_n z_n}{x_n x'_n}
\end{aligned}$$

and subsequently

- (a) $x_{n-1} x'_n = x_n x'_{n+1}$
- (b) $\frac{y'_{n+1}}{x'_{n+1}} = \frac{y'_n}{x'_n}$
- (c) $\frac{y_n}{x_n} = \frac{y_{n-1}}{x_{n-1}}$
- (d) $\frac{z'_{n+1}}{x_n x'_{n+1}} + \frac{z_n}{x_n} - \frac{y_n z'_n}{x_n x'_n} = \frac{z_{n-1}}{x'_n x_{n-1}} + \frac{z'_n}{x'_n} - \frac{y'_n z_n}{x_n x'_n}.$

Relations $C4$ and $C5$ give the same conditions (a)–(d).

- (a) $|a| = |b| - 1 = |c| - 1 = n.$
- (b) $|d| = n.$

The relation $C3$ can be reduced to the following equation:

$$(F_a|_n F_b)|_n (F_c|_{(n+1)'} F_d) = (F_a|_n F_c)|_n (F_b|_{(n+1)'} F_d).$$

After simple calculations we find out that the above equality is equivalent to:

$$2. \frac{y_n}{x_n} = \frac{y'_{n+1}}{x'_{n+1}}$$

Relations $C4$ and $C5$ can be reduced to the condition (e) as well.

- (a) $|d| = n + 2.$

In this case relations $C3 - C5$ are reduced to the condition (c).

$$3. |a| = |b| + 1 = |c| + 1 = n.$$

- (a) $|d| = n - 2.$
- (b) $|d| = n.$

After simple calculations we find out that in the cases 3(i) and 3(ii) the relations $C3 - C5$ follow from the conditions (c) and (e).

The conditions (a)–(e) are equivalent to conditions satisfied by x'_i, y_i, y'_i and z'_i in Example III.4.5. Therefore we have proved that \mathcal{A} is a geometrically sufficient partial Conway algebra.

The partial algebra discussed above in Example III.4.5 defines an invariant of links, the second component of which is a polynomial of infinite number of variables. It is natural to ask how is this polynomial related to other known invariants of links and whether it generalizes the Jones-Conway polynomial.

Problem III.4.6

1. *Do there exist two oriented links which have the same Jones-Conway polynomial but they can be distinguished by a polynomial of infinitely many variables?*¹⁵
2. *Do there exist two algebraically equivalent links (i.e. links which have always the same invariants coming from Conway algebras) which can be distinguished by a polynomial of infinite variables?*

It is quite possible that our partial algebra of polynomials of infinite number of variables can be extended to a Conway algebra (in that case Traczyk result would follow from Toyoda Theorem [Toy-1, Toy-2, Toy-3, Toy-4]) At any rate, Birman's examples [Bi-2] cannot be distinguished one from the other by a polynomial of infinite variables. In particular we have:

Exercise III.4.7 *Prove that the links from Fig. 3.21 have the same polynomials of infinite variables. The same for links pictured at Fig. 3.22.*

The next example of a geometrically sufficient partial Conway algebra defines important classical invariants of links: classical signature, Tristram-Levine signature and its generalization — supersignature (see [P-T-3]; we were unable to prove there an existence of a supersignature, which would solve Milnor's unknotting conjecture. The dream of “supersignature” was realized with invention of Khovanov homology (see Chapter X) and Rasmussen s-signature [Kh-1, Ras]). An advantage of signature (especially in

¹⁵Adam Sikora (then a student of P. Traczyk) proved that the answer for the question 4.6.1 is negative, [Si-1]. If the coefficient of w_0 in equations (1)–(i') was not equal to 1 but was non-invertible then the polynomial of infinite number of variables would distinguish some of Birman's links which can not be distinguished by the Jones-Conway polynomial; [P-6]

view of the case of polynomial of infinite variables which we have just discussed) is the existence of examples of algebraically equivalent links which can be distinguished by the signature (e.g. knot pictured at Fig. 3.21). This implies that there exist algebraically equivalent links which are not skein equivalent. It is somewhat uncomfortable, though, that until now we know no purely combinatorial proof that signature and supersignature are invariants of ambient isotopy classes of links (compare Chapter IV).

Definition III.4.8 (*Supersignature Algebra*)

For a pair of real number u, v , such that $u \cdot v > 0$, we define a partial Conway algebra $\mathcal{A}_{u,v}$ which we call the supersignature algebra. We define it as follows: $A = (R \cup iR) \times (\mathbb{Z} \cup \infty)$, $B_1 = B_2 = \{(r_1, z_1), (r_2, z_2) \in A \times A \mid \text{if } r_1 \in R \text{ then } r_2 \in iR, \text{ if } r_1 \in iR \text{ then } r_2 \in R, \text{ if } z_1, z_2 \neq \infty \text{ then } |z_1 - z_2| = 1, \text{ and } r_i = 0 \text{ if and only if } z_i = \infty\}$.

The operations $|$ and \star are defined as follows:

The first coordinates of elements in $A = (R \cup iR) \times (\mathbb{Z} \cup \infty)$ are related by the equality

$$-uw_1 + vw_2 = iw_0, \text{ where } w_1, w_2, w_0 \in R \cup iR$$

which is similar to the case of the Jones-Conway polynomial (it is enough to substitute $u = -xi$ and $v = yi$ to get the equation III.1.9). In particular, the first coordinate of the result of an operation depends only on the first coordinates of arguments, so we will write simply $w_1 = w_2|w_0$ and $w_2 = w_1 \star w_0$.

In order to describe the second coordinate of the result of operations $|$ and \star we write $(r, z) = (r_1, z_1)|(r_0, z_0)$ or $(r_1, z_1) \star (r_0, z_0)$ where z is determined by the following conditions:

1. $i^z = \frac{r}{|r|}$ if $r \neq 0$,
2. $|z - z_0| = 1$ if $r \neq 0$ and $r_0 \neq 0$,
3. $z = z_1$ if $r_0 = 0$,
4. $z = \infty$ if $r = 0$.

The 0-argument operations (constants or unary operations) a_i are defined as follows:

$$a_1 = (1, 0), \dots, a_k = \left(\left(\frac{v-u}{i} \right)^{k-1}, \begin{cases} -(k-1) & \text{if } u < v \\ \infty & \text{if } u = v \\ k-1 & \text{if } u > v \end{cases} \right), \dots$$

And finally, D is the subset of $A \times A \times A \times A$ consisting of elements for which the relations $C3 - C5$ are satisfied.

Problem III.4.9 ([P-T-2]) For which values of (u, v) is $\mathcal{A}_{u,v}$ a geometrically sufficient partial Conway algebra?

For the pair u, v for which the answer is positive, the algebra $\mathcal{A}_{u,v}$ defines an invariant of links, the second component of which we call the supersignature and we denote it by $\delta_{u,v}(L)$.

In Chapter IV we show that the answer for Problem III.4.9 is affirmative for $u = v \in (-\infty, -\frac{1}{2}] \cup [\frac{1}{2}, \infty)$ and then for such u and v the supersignature coincides with Tristram-Levin signature [Tr, Le, Gor] (unless it is equal ∞). In particular, for $u = v = \frac{1}{2}$ we get the classical signature.¹⁶ The proof of this fact goes beyond simple purely combinatorial methods and therefore we postpone it until the next chapter. Now, let us look what the obstructions for a direct solution of Problem III.4.9 are.

1. Suppose that the condition (i) in Definition III.4.2 is satisfied. This implies that, given any resolving tree of a link, all operations needed to compute the value of the invariant in the root are admissible. This is because the first coordinate of elements of the supersignature algebra (which is known to be an invariant of links as it is a variant of the Jones-Conway polynomial) assumes real values for links with an odd number of components and purely imaginary values for links with an even number of components.
2. We would like to prove the condition (ii) of Definition III.4.2. The condition asserts that different resolving trees of a given link yield the same value in their roots (equivalently, conditions $C3 - C5$ are satisfied when needed). The first attempt of the proof is to check whether the condition $C3$ is true whenever both sides of the equality are well defined (this is enough because of Lemma III.4.4) — the idea is similar to Example III.4.5. This time, however, this is not the case, as we see from the following example.

¹⁶T.Przytycka found (in the Spring of 1985) the values of u, v , for which the supersignature is not well defined. In her examples $u \neq v, v^{-1}$. P. Traczyk and M. Wiśniewska [Wis] have found a neighborhood of the origin on the plane consisting of pairs (u, v) for which the supersignature does not exist (Problem III.4.9 has negative answer). There is a possibility that there exists a supersignature related to the Jones polynomial, i.e. defined for pairs (u, v) such that $v = \frac{-t}{\sqrt{t}-\sqrt{-t}} = t^2 u$ where t is a negative real number.

Example III.4.10 *Let us consider the condition C3 for a supersignature algebra $\mathcal{A}_{u,v}$*

$$((r_a, z_a)|(r_b, z_b))((r_c, z_c)|(r_d, z_d)) = ((r_a, z_a)|(r_c, z_c))((r_b, z_b)|(r_d, z_d))$$

Because of the definition of the operation $|$ we get:

1. $r_a|r_b = \frac{1}{u}(ir_b + vr_a)$
2. $r_a|r_c = \frac{1}{u}(ir_c + vr_a)$
3. $r_c|r_d = \frac{1}{u}(ir_d + vr_c)$
4. $r_b|r_d = \frac{1}{u}(ir_d + vr_b)$
5. $(r_a|r_b)|(r_c|r_d) = (r_a|r_c)|(r_b|r_d) = \frac{1}{u^2}d + ivr_c + ivr_b + v^2r_a$

Now suppose that

6. $u, v > 0; r_a, r_d \in R; r_b, r_c \in iR; z_a = -2, z_b = z_c = -1, z_d = 0.$

Because of these properties and in view of the condition (1) of Definition III.4.8) we get:

7. $r_a < 0, r_d > 0, ir_b > 0, ir_c > 0$

Further, suppose that

8. $ir_b + vr_a > 0, ir_c + vr_a < 0, m - r_d + ivr_c < 0, -r_d + ivr_b > 0$

and

9. $-r_d + ivr_c + ivr_b + v^2r_a < 0.$

The conditions 6–10 may be satisfied (even for $u = v = \frac{1}{2}$), yet the value of the second coordinate computed in left-hand-side of the condition C3 is equal 2, while on the right-hand-side we get -2.

Exercise III.4.11 *If, for some values of u, v , the answer for Problem III.4.9 is affirmative then, in view of the Example III.4.10, we can use the algebra to find bounds on Jones-Conway polynomials of links. Find these bounds.*

Lemma III.4.12 *The supersignature $\delta_{u,v}$ (for these u, v for which it exists) satisfies the following conditions:*

$$(a) \quad \delta_{u,v}(L) = -\delta_{v,u}(\overline{L}),$$

$$(b) \delta_{u,v}(L_1 \# L_2) = \delta_{u,v}(L_1) + \delta_{u,v}(L_2)$$

$$(c) \delta_{u,v}(L_1 \sqcup L_2) = \delta_{u,v}(L_1) + \delta_{u,v}(L_2) + \varepsilon(u, v) \text{ where}$$

$$\varepsilon(u, v) = \begin{cases} 1 & \text{if } u > v \\ \infty & \text{if } u = v \\ -1 & \text{if } u < v \end{cases}$$

$$(d) \delta_{u,v}(L_+) \leq \delta_{u,v}(L_-) \text{ if } \delta_{u,v}(L_+) \neq \infty \text{ and } u, v > 0.$$

Proof:

In the proof of conditions (a), (b) and (d) we use a standard induction with respect to the number of crossings for a choice of base points in the diagram. Moreover in the proof of (a) we apply Lemma III.3.21 which implies equality

$$r_L(u, v) = r_{\overline{L}}(-v, -u) = \begin{cases} r_{\overline{L}}(v, u) & \text{if } L \text{ has an odd number of components} \\ -r_{\overline{L}}(v, u) & \text{if } L \text{ has an even number of components} \end{cases}$$

and in the proof of (b) we apply Corollary III.3.27 which implies

$$r_{L_1 \# L_2}(u, v) = r_{L_1}(u, v) \cdot r_{L_2}(u, v).$$

The condition (c) follows from (b) once we note that $L_1 \sqcup L_2$ can be obtained as a connected sum $(L_1 \# T_2) \# L_2$ where T_2 is a trivial link with two components (Fig. 4.2) and $\varepsilon(u, v) = \delta_{u,v}(T_2)$.

□



$$(L_1 \# T_2) \# L_2 = L_1 \sqcup L_2$$

Fig. 4.2

Exercise III.4.13 Show that knots 8_8 and $\overline{8}_8$ (Fig. 3.11) can be distinguished by the supersignature $\delta_{u,v}$ for some values of u and v ; however if $u = v$ then the supersignature of both knots is equal to zero.

Sketch of the argument.

1.

$$r_{8_8}(u, v) = -uv^{-1} + 2 + v^{-2} + u^{-1}v - 2u^{-1}v^{-1} - u^{-2}v^2 - 2u^{-2} + u^{-2}v^{-2} + u^{-3}v + u^{-3}v^{-1}$$

Hence we can compute that:

- (a) $r_{8_8}(u, u) > 0$
- (b) $r_{8_8}(u, v) < 0$ if $u \gg v \approx c$ or $v \gg u \approx c$
- (c) $r_{8_8}(u, v) > 0$ if $u \ll v \approx c$ or $v \ll u \approx c$,

where c is a constant number.

2. The removal (smoothing) of a crossing in some diagram of the knot 8_8 yields a trivial knot of two components (c.f. Example III.3.19).

In view of the definition of the supersignature, the above observations imply that for $u, v > 0$ we have:

$$\sigma_{u,v}(\overline{8_8}) = \begin{cases} 0 & \text{if } u \ll v \\ 0 & \text{if } u = v \\ -2 & \text{if } u \gg v \end{cases}$$

Now applying Lemma III.4.12(a) we show easily the values of u and v for which $\sigma_{u,v}(8_8) \neq \sigma_{u,v}(\overline{8_8})$; see Fig. 4.3.

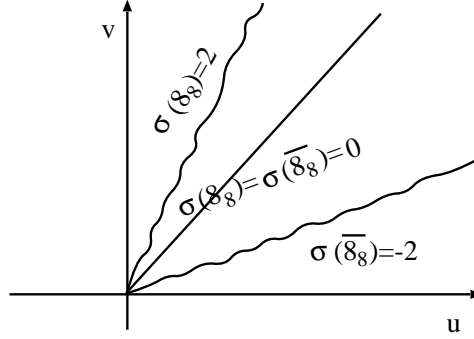


Fig. 4.3

Remark III.4.14 An important problem in knot theory concerns so-called unknotting (or Gordian) number of a link which — by definition — is the minimal number of tunnel-to-bridges changes which have to be done to modify a given link to the trivial one (see Chapter IV). The supersignature may seem to be very useful to bound the unknotting number since its values for L_+ and L_- differ by at most 2, unless one of them is ∞ . However, ∞ was assigned for our convenience only, i.e. to simplify the description. One may try to find another value of $\sigma_{u,v}(L)$ (different from ∞) for links with $r_L(u, v) = 0$. Useful information can be found in Lemma III.3.38(ii) which reveals that

the Jones-Conway polynomial, and thus also the polynomial $r_L(u, v)$, is not identically zero. Thus, even if for some u_0 and v_0 we have $r_L(u_0, v_0) = 0$, then in some neighborhood of (u_0, v_0) the polynomial $r_L(u, v)$ assumes non-zero values. Subsequently, one may try to change the value of $\sigma_{u_0, v_0}(L)$ replacing ∞ with the mean value of $\sigma_{u, v} \neq 0$. We leave this idea to the reader as a research problem.

Exercise III.4.15 Prove that, if for a pair (u, v) there exists the supersignature $\sigma_{u, v}(L)$ and it is not equal ∞ , then the minimal depth of the resolving tree of L (i.e. the distance from the root to the farthest leaf) is not smaller than

$$\frac{|\sigma_{u, v}(L)|}{2} - \varepsilon(L),$$

where

$$\varepsilon = \begin{cases} 0 & \text{if } u = v \\ n(L) - 1 & \text{if } u \neq v \end{cases};$$

and $n(L)$ denotes the number of components of L .

Example III.4.16 The equivalence classes of the relation \sim_c of oriented links make geometrically sufficient partial Conway algebra. Elements a_i of this algebra are classes of trivial links with i components, the operation $|$ (resp. \star) is defined on a pair of classes of links if they can be represented by diagrams of type L_- and L_0 (resp. L_+ and L_0) and the result is equal to the class of L_+ (resp. L_-). The definition of \sim_c equivalence is chosen in such a way that axioms C1 – C7 are satisfied when it is needed. Let us note that this partial algebra is a universal geometrically sufficient partial Conway algebra, that is there exists the unique homomorphism of this algebra onto any geometrically sufficient partial Conway algebra.

As we have mentioned in Remark III.3.23, some Conway algebras admit an operation \circ which allows to find the value of an invariant for L_0 , provided we know its value for L_+ and L_- . Geometrically sufficient partial Conway algebras described in III.4.5 and III.4.8 allow for such an operation, it is an open question, however, whether the universal Conway algebra admits such an operation. More precisely, it is an open question whether the equations $a|x = b$ and $a \star x = b$ can have at most one solution — see Problem III.3.30).

The involution τ from Lemma III.3.32 is realized in the universal geometrically sufficient partial Conway algebra as the operation of replacing a given \sim_c -equivalence class by the class of its mirror image.

Exercise III.4.17

1. Suppose that τ is an involution of a geometrically sufficient partial Conway algebra. What properties have to be satisfied τ in order to satisfy the condition

$$\tau(w_L) = w_{\overline{L}}?$$

2. Find such an involution τ for the algebra from Example III.4.5.
3. Prove that there exists no such involution for the algebra of the super-signature $\sigma_{u,v}(u \neq v)$ but if we modify the algebra so that new elements are quadruples $(r_L(u, v), r_L(v, u), \sigma_{u,v}(L), \sigma_{v,u}(L))$, then the involution τ can be defined (c.f. Lemma III.4.12(a)).

Also Lemma III.3.28, which allows to find out the invariant of the connected sum and disjoint union of links, can be partially extended to the case of geometrically sufficient partial Conway algebras (c.f. Lemma III.4.12(b) and (c)).

III.5 Kauffman approach

It is a natural question to ask whether the three diagrams, L_+ , L_- , and L_0 , which have been used to build Conway type invariants can be replaced by other diagrams. In fact, at the turn of December and January of 1984/85, Krzysztof Nowiński suggested considering another diagram apart from L_+ , L_- and L_0 , namely, the diagram obtained by smoothening L_+ without preserving the orientation of L_+ (Fig. 5.1) — at that time, however, we did not make any effort to exploit this idea to get new invariants of links; likely we were discouraged by a lack of a natural orientation on L_∞ .

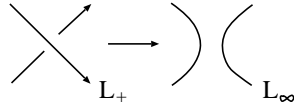


Fig. 5.1

In the early Spring of 1985 R.Brandt, W.B.R.Lickorish i K.C.Millett [B-L-M] and, independently, C.F.Ho [Ho] proved that four diagrams of un-oriented links pictured in Fig. 5.2 (the $+$ sign in L_+ does not denote the sign of the crossing, even if the sign is defined) can be used to construct

invariants of unoriented links.



Fig. 5.2

Theorem III.5.1 *There exists a uniquely determined invariant Q which to any (ambient) isotopy class of invariant links associates an element of $\mathbb{Z}[x^{\pm 1}]$. The invariant Q satisfies the following conditions:*

- (1) *if T_1 is the trivial knot then $Q_{T_1}(x) = 1$*
- (2) *$Q_{L_+}(x) + Q_{L_-}(x) = x(Q_{L_0}(x) + Q_{L_\infty}(x))$, where unoriented diagrams of links L_+, L_-, L_0 and L_∞ are identical outside of the parts pictured in Fig. 5.2.*

The proof of Theorem 5.1 is similar to that of Theorem III.1.2 (c.f. [B-L-M]). We will discuss it later in a more general context.

The polynomial $Q_L(x)$ has a number of features similar to these of the Homflypt (Jones-Conway) polynomial $P_L(x, y)$. The proof of these properties is left for the reader as an exercise

Exercise III.5.2 *Prove that:*

- (a) $Q_{T_n}(x) = (\frac{2-x}{x})^{n-1}$. where T_n is the trivial link of n components, $[(b)]$
 $Q_{L_1 \# L_2}(x) = Q_{L_1}(x)Q_{L_2}(x)$,
- (c) $Q_{L_1 \sqcup L_2} = \mu Q_{L_1}(x)Q_{L_2}(x)$, where $\mu = Q_{T_2}(x) = 2x^{-1} - 1$ trivial link.
- (d) $Q_L(x) = Q_{\bar{L}}(x)$, where \bar{L} is the mirror image of the link L .
- (e) $Q_L(x) = Q_{m(L)}(x)$, where $m(L)$ is the mutant of the link L .
- (f) $Q_L(x)(-2) = (-2)^{com(L)-1}$.

The polynomial $Q_L(x)$ can sometime distinguish links which are \sim_c equivalent

Exercise III.5.3 (a) *Show that $Q_L(x)$ distinguishes knots $8_8, 10_{129}$, and 13_{6714} (c.f. Example 3.19).*

- (b) *Conclude that the above knots can not be obtained by mutation of one another.*

Now we will discuss Kauffman approach [K-4, K-5] which, in particular, allows to generalize the polynomial $Q_L(x)$ to an invariant distinguishing mirror images. This approach is based on the Kauffman's idea that instead of considering diagrams modulo equivalence (that is diagrams up to all three Reidemeister moves) one can consider diagrams up to the equivalence relation which is based on second and third Reidemeister moves. This way one does not get an invariant of links but often a simple correction/balancing allows to construct a true invariant¹⁷.

Definition III.5.4 *Two diagrams are called regularly isotopic if one can be obtained from the other via a finite sequence of Reidemeister moves of types two and three. The definition of regular isotopy makes sense for orientable as well as for unorientable diagrams.*

While working with regular isotopy invariants one is able to take into account some properties of diagrams which are not preserved by Reidemeister moves of the first type.

Lemma III.5.5 *Let the writhe number $w(L)$ of an oriented link diagram L be defined by $w(L) = \sum_i \text{sgn}(p_i)$, where the sum is taken over all crossings of L . Then $w(L)$ is regular isotopy invariant and $w(-L) = w(L) = -w(\bar{L})$. The number $w(L)$ is also called the Tait¹⁸ number of the diagram of the link L and denoted by $\text{Tait}(L)$. Sometimes this number is also called a twist number of the diagram and denoted by $\text{tw}(L)$.*

Proof: It is enough to show that the second Reidemeister move cancels or produces two crossings with opposite signs and the third Reidemeister move preserves signs of all crossing. Moreover, we note that passing from the diagram L to the diagram with the opposite orientation $-L$ does not change the sign of a crossing and passing from L to its mirror image \bar{L} changes the sign of all crossings. \square

¹⁷Kauffman approach has a good interpretation in terms of framed links (the approach is used in Chapter X on skein modules of 3-dimensional manifolds).

¹⁸Peter Guthrie Tait (1831-1901) was a Scottish physicist who, influenced by vortex theory of atoms by W. Thompson (Lord Kelvin), was tabulating diagrams of links. The number of crossings and the number $w(L)$ were important "invariants" of this tabulation.

The idea of Kauffman is based on the observation that the trivial knot can be represented (up to ambient isotopy) by different regular isotopy classes of diagrams and to any such a class we can associate different values of some invariant. To any diagram T_1 representing the trivial knot, Kauffman associates the monomial $a^{w(T_1)}$. Subsequently, the Kauffman definition of invariants is similar to the definition of Conway and Jones polynomials, $P_L(x, y)$ and $Q_L(x)$. While passing from regular isotopy invariants to isotopy invariants, Kauffman applies the following fact.

Lemma III.5.6 *Let us consider the following elementary move on a diagram of a link, denoted by $(R_{0,5}^{\pm 1})$ and called the first weakened (or balanced) Reidemeister move. That is the move which allows to create or to cancel the pair of curls of the opposite signs, see Fig. 5.3. Let us observe that signs of crossings in curls do not depend on the orientation of the diagram.*

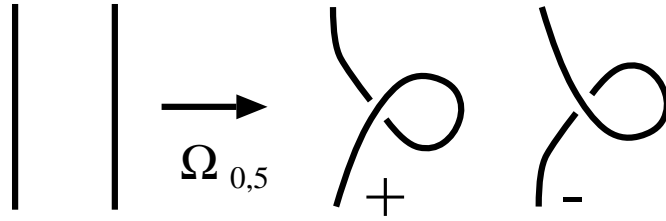


Fig. 5.3

Then one can obtain a diagram L_1 from another diagram L_2 by regular isotopy and first weakened Reidemeister move if and only if L_1 is isotopic to L_2 and $w(L_1) = w(L_2)$.

The proof of III.5.6 becomes clear once we note that the move $R_{0,5}^{\pm 1}$ enables us to carry a twist to any place in the diagram.

Now we show how, using the Kauffman approach, that the Conway polynomial can be generalized to the Homflypt polynomial and the polynomial Q_L can be modified to another invariant which we will call Kauffman polynomial (or Kauffman polynomial of two variables).

Theorem III.5.7 ([K-4])

1. *There exists a uniquely defined invariant of regular isotopy of oriented diagrams, denoted by $(R_L(a, z))$, which is a polynomial in $\mathbb{Z}[a^{\pm 1}, z^{\pm 1}]$ and which satisfies the following conditions:*

(a) $R_{T_1}(a, z) = a^{w(T_1)}$, where T_1 is a diagram of a knot isotopic to the trivial knot.

(b) $R_{L_+}(a, z) - R_{L_-}(a, z) = zR_{L_0}(a, z)$.

2. *For any diagram L we consider a polynomial $G_L(a, z) = a^{-w(L)}R_L(a, z)$. Then $G_L(a, z)$ is an invariant of ambient isotopy of oriented links and it is equivalent to the Homflypt polynomial, that is*

$$G_L(a, z) = P_L(x, y) \text{ where } x = \frac{a}{z}, y = \frac{-1}{za}.$$

Proof. The method of the proof of Theorem III.1.2 can be used to prove III.5.7 but we will present a proof based on the existence of the Homflypt polynomial, since its existence was already proved. Namely, we consider a substitution in $P_L(x, y)$ by setting $x = \frac{a}{z}$ and $y = \frac{-1}{az}$. As the result we obtain a polynomial invariant of ambient isotopy classes of oriented links which we call $\tilde{G}_L(a, z)$ and which satisfies the following conditions:

(a) $\tilde{G}_{T_1}(a, z) = 1$

(b) $a\tilde{G}_{L_+}(a, z) - \frac{1}{a}\tilde{G}_{L_-}(a, z) = z\tilde{G}_{L_0}(a, z)$.

Now to any oriented diagram we associate a polynomial $\tilde{R}_L(a, z) = a^{w(L)}\tilde{G}_L(a, z)$. It is easy to see that the first Reidemeister move changes the value of $\tilde{R}_L(a, z)$ according to the sign of the curl:

$$\tilde{R}_{\text{curl}}(a, z) = a\tilde{R}_{\text{curl}}(a, z)$$

$$\tilde{R}_{\text{curl}}(a, z) = a^{-1}\tilde{R}_{\text{curl}}(a, z)$$

Moreover the second and the third Reidemeister move do not change $\tilde{R}_L(a, z)$. Thus $\tilde{R}_L(a, z)$ is an invariant of regular isotopy of oriented diagrams and it satisfies the condition $\tilde{R}_{T_1}(a, z) = a^{w(T_1)}$. It is also easy to verify that $\tilde{R}_L(a, z)$ satisfies the condition (b)) of Theorem III.5.7. Moreover,

since any diagram has a resolving tree therefore the polynomial $R_L(a, z)$, if it exists, is uniquely determined by the conditions (a) and (b). Thus, setting $R_L(a, z) = \tilde{R}_L(a, z)$ and $G_L(a, z) = \tilde{G}_L(a, z)$, we complete the proof of Theorem III.5.7.

Theorem III.5.8 ([K-5])

- (1) *There exists a uniquely defined invariant Λ which to any regular isotopy class attaches a polynomial in $\mathbb{Z}[a^{\pm 1}, z^{\pm 1}]$ and which satisfies the following conditions:*
- (a) $\Lambda_{T_1}(a, z) = a^{w(T_1)}$,
 - (b) $\Lambda_{L_+} + \Lambda_{L_-} = z(\Lambda_{L_0} + \Lambda_{L_\infty})$.
- (2) *For any oriented diagram D we define $F_D(a, z) = a^{w(D)} \Lambda_D(a, z)$. Then $F_D(a, z)$ is an invariant of (ambient) isotopy classes of oriented links and it is a generalization of the polynomial Q , that is $Q_L(x) = F_L(1, x)$.*

Proof. Part (1) of the theorem will be proved later in a more general context.

Part (2) follows from (1) if we notice that the first Reidemeister move changes the polynomial $\Lambda_L(a, z)$ in the following way:

$$\Lambda_{\text{cross}}(a, z) = a \Lambda_{\text{smooth}}(a, z) \quad \text{and} \quad \Lambda_{\text{smooth}}(a, z) = a^{-1} \Lambda_{\text{cross}}(a, z)$$

The polynomial $F_L(a, z)$ is called the Kauffman polynomial of the link L .

Now we will present some elementary properties of Kauffman polynomial.

Theorem III.5.9

- (a) $F_{T_n} = \left(\frac{a+a^{-1}-z}{z}\right)^{n-1}$,
- (b) $F_{L_1 \# L_2}(a, z) = F_{L_1}(a, z) \cdot F_{L_2}(a, z)$.
- (c) $F_{L_1 \sqcup L_2}(a, z) = \mu F_{L_1}(a, z) \cdot F_{L_2}(a, z)$ where $\mu = F_{T_2} = \frac{a^{-1}+a}{z} - 1$ is the value of the invariant on the trivial link with two components.
- (d) $F_L(a, z) = F_{-L}(a, z)$.

$$(e) F_{\overline{L}}(a, z) = F_L(a^{-1}, z).$$

$$(f) F_L(a, z) = F_{m(L)}(a, z) \text{ where } m(L) \text{ is a mutant of the link } L.$$

The proof is rather straightforward, compare it with Lemma III.3.21, Theorem III.3.25, Corollary III.3.27, Theorem III.3.9, and Exercise III.5.2).

The polynomial Λ does not depend on the orientation of the diagram D . Therefore the dependence of the Kauffman polynomial F on the orientation of components of D is easy to describe — this is because $F_D(a, z)$ differs from $\Lambda_D(a, z)$ by a power of a .

Lemma III.5.10 *Let $D = \{D_1, D_2, \dots, D_i, \dots, D_n\}$ be a diagram of an oriented link with n components and let $D' := \{D_1, D_2, \dots, -D_i, \dots, D_n\}$. We set $\lambda_i = \frac{1}{2} \sum \text{sgn } p_j$ where the summation is over all crossings of D_i with $D - D_i$. (The number λ_i is called the linking number of D_i and $D - D_i$ and it is often denoted by $\text{lk}(D_i, D - D_i)$; it is an invariant of ambient isotopy of D (D_i should be kept as i component); this follows easily from considering Reidemeister moves. Then*

$$F_{D'}(a, z) = a^{-w(D')} a^{w(D)} F_D(a, z) = a^{4\lambda_i} F_D(a, z).$$

Let us note that the above lemma implies that the Kauffman polynomial gives only as much information about orientations of D as comes from the linking numbers of its components D_i with complements $D - D_i$ (we coded this information in weighted simplex of global linking numbers; Definition III.3.44 and Exercise III.3.46).

The Kaufman polynomial is much more useful for distinguishing a given link from its mirror image (achirality of a link). However, we gave already an example of a (pretzel) link of two components, which is a mutant of its mirror image and they are not isotopic (Fig. 3.7), although Kauffman expected initially that such a situation is impossible for knots.

Conjecture III.5.11 *[[K-5].] If a knot K is not isotopic to its mirror image (\overline{K} or $-\overline{K}$) then $F_K(a, z) \neq F_{\overline{K}}(a, z)$.*

The knots 9_{42} and 10_{71} (according to the Rolfsen's notation [Ro-1], see Fig. 5.4) contradict the conjecture, since they are isotopic to their mirror images still they have the same Kauffman polynomials as their mirror images.

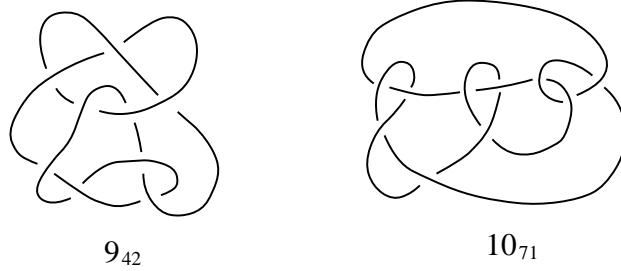


Fig. 5.4

Remark III.5.12 *It was an open problem for a while whether the knot 9_{42} is algebraically equivalent to $\overline{9}_{42}$? (It is easy to find out that $\sigma(9_{42}) = -2 = -\sigma(\overline{9}_{42})$ and therefore the two knots are not \sim_c equivalent). However from Sikora result [Si-2] it follows that no invariant coming from a Conway algebra can distinguish 9_{42} from $\overline{9}_{42}$.*

The Kauffman polynomial is also a generalization of the Jones polynomial.

Theorem III.5.13 ([Li-2])

$$V_L(t) = F_L(t^{3/4}, -(t^{-1/4} + t^{1/4}))$$

Proof. Let us begin by characterizing the Kauffman polynomial without using regular isotopy.

Lemma III.5.14 *The Kauffman polynomial is uniquely determined by the following conditions:*

- (1) $F_{T_1}(a, z) = 1$,
- (2) Consider the Conway skein triple of oriented link diagrams, L_+ , L_- , and L_0 of Figure 1.1. Let $cp(L)$ denote the number of components of the link L . We consider two cases dependent on whether the modified crossing in L_+ is a selfcrossing or the mixed one.

(i) $cp(L_+) = cp(L_0) - 1$ (a selfcrossing case):

$$aF_{L_+}(a, z) + \frac{1}{a}F_{L_-}(a, z) = z(F_{L_0}(a, z) + a^{-4\lambda}F_{L_\infty}(a, z)),$$

where $cp(L_\infty) = cp(L_+)$ and L_∞ is given any one of two possible orientations and $\lambda = lk(L_i, L_0 - L_i)$ with L_i being the new component of L_0 , the orientation of which does not agree with the orientation of the corresponding component of L_∞ .

(ii) $cp(L_+) = cp(L_0) + 1$ (a mixed crossing case):

$$aF_{L_+}(a, z) + \frac{1}{a}F_{L_-}(a, z) = z(F_{L_0}(a, z) + a^{-4\lambda+2}F_{L_\infty}(a, z)),$$

where $cp(L_\infty) = cp(L_0) = cp(L_+) - 1$ and L_∞ is given any one of two possible orientations and $\lambda = lk(L_i, L_+ - L_i)$ with L_i being the component of L_+ , the orientation of which does not agree with the orientation of the corresponding component of L_∞ .

Proof: In view of the definition of $F_L(a, z)$, Lemma III.5.14 follows easily from Lemma III.5.10. We show the calculation, as an example, in the case 2(ii). By the definition we have;

$$\Lambda_{L_+} + \Lambda_{L_-} = z(\Lambda_{L_0} + \Lambda_{L_\infty}), \text{ therefore}$$

$$a^{w(L_+)}F_{L_+} + a^{w(L_-)}F_{L_-} = z(a^{w(L_0)}F_{L_0} + a^{w(L_\infty)}F_{L_\infty}) \text{ so:}$$

$$aF_{L_+} + a^{-1}F_{L_-} = z(F_{L_0} + a^{w(L_\infty)-w(L_0)}F_{L_\infty})$$

which reduces to the formula of 2(ii) as $w(L_\infty) - w(L_0) = 2 - 2lk(L_i, L_+ - L_i)$.

□

Proceeding with the proof of Theorem III.5.13 we need an additional characterization of the Jones polynomial. Let us recall that the Jones polynomial is uniquely defined by the following conditions:

1. $V_{T_1}(t) = 1$,
2. $\frac{1}{t}V_{L_+}(t) - tV_{L_-}(t) = (\sqrt{t} - \frac{1}{\sqrt{t}})V_{L_0}(t)$.

Lemma III.5.15 (Jones reversing formula) *Suppose that L_i is a component of an oriented link L and $\lambda = lk(L_i, L - L_i)$. If L' is a link obtained from L by reversing the orientation of the component L_i then $V_{L'}(t) = t^{-3\lambda}V_L(t)$.*

Notice that the Jones reversing formula generalizes immediately to the case when we reverse the orientation of a sublink U of a link L , that is having $\lambda = lk(U, L - U)$ we obtain $V_{L'}(t) = t^{-3\lambda}V_L(t)$ where L' is obtained from L by reversing orientation of U .

Proof of Lemma III.5.15. We present the original proof by Lickorish and Millett [L-M-3]. Very simple proof was found by L.Kauffman (using Kauffman bracket polynomial).

The proof consists of five steps.

- (1) Simple calculations show that Lemma III.5.15 is true for links pictured in Fig. 5.5.



Fig. 5.5

- (2) $V_L(t) = V_{-L}(t)$ and moreover, since $V_{L_1 \# L_2}(t) = V_{L_1}(t) \cdot V_{L_2}(t)$ and $V_{\bar{L}}(t) = V_L(t^{-1})$, thus if Lemma III.5.15 is true for two links L_1 and L_2 then it is true also for \bar{L}_1 , $L_1 \# L_2$ and $L_1 \# -L_2$.
- (3) Fix a diagram of L . A standard induction with respect to the number of the (self)crossings of $L_i \subset L$ and with respect to the number of bad (self)crossings of L_i , implies that we can assume that L_i is a descending diagram and thus, in particular, L_i is a diagram of the trivial knot.
- (4) Let us consider a 2-dimensional disc D in S^3 , the boundary of which is the knot L_i . We may assume that $L - L_i$ meets D transversally in a finite number of points, say in n points. Now we proceed by using induction with respect to the number n . The initial conditions of the induction will be discussed later in the step (5). Now we assume that $n \geq 4$. Figure 5.6 presents Conway's skein triple of diagrams, L_+ , L_- , and L_0 , where L_0 represents L_i . The disc bounded by L_i is cut by $L - L_i$ in points which are marked by crosses. The knot L_i becomes in L_- a trivial link with two components γ_1^- and γ_2^- which bound discs meeting the remainder of L_- in n_1 and n_2 points, respectively. Besides, the linking number of γ_1^- and γ_2^- with the remainder of the link L_- is λ_1 and λ_2 , respectively. In the case of L_+ the situation is similar, only this time γ_1^+ and γ_2^+ are linked, i.e. $\text{lk}(\gamma_1^+, \gamma_2^+) = 1$, see Fig. 5.6.

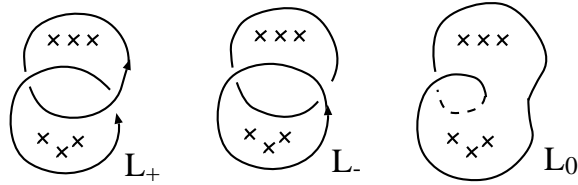


Fig. 5.6

Now, we have $n_1 + n_2 = n$ and $\lambda_1 + \lambda_2 = \lambda$. We may assume that both n_1 and n_2 are 2 at least. Now, let L'_+ , L'_- , L'_0 be links obtained from links L_+ , L_- , L_0 , respectively, by changing the orientation of components (γ_1^+, γ_2^+) , (γ_1^-, γ_2^-) and L_i , respectively. Then

$$\frac{1}{t}V_{L_+}(t) - tV_{L_-}(t) = (\sqrt{t} - \frac{1}{\sqrt{t}})V_{L_0}(t)$$

and

$$\frac{1}{t}V_{L'_+}(t) - tV_{L'_-}(t) = (\sqrt{t} - \frac{1}{\sqrt{t}})V_{L'_0}(t).$$

The inductive assumption allows to apply Lemma III.5.15 to L_+ and L_- . Changing the orientation of γ_1^+ and γ_2^+ in L_+ we obtain:

$$V_{L'_+}(t) = t^{-3(\lambda_1 + \lambda_2)}V_{L_+}(t)$$

while changing the orientation of γ_1^- and γ_2^+ in L_- we similarly get:

$$V_{L'_-}(t) = t^{-3(\lambda_1 + \lambda_2)}V_{L_-}(t).$$

Therefore we get $t^{-3\lambda}V_L(t) = V_{L'}(t)$ as needed.

If $n < 4$ then there are problems with the inductive assumption for L_+ . However, we can extend the above argument for the case $n = 3$, $\lambda = \pm 3$, provided that the lemma is true for $n \leq 2$ and for $n = 3$, $\lambda = \pm 1$.

(5) Now let us consider the case of $n = 3$, $\lambda = \pm 1$.

We are to show that for the link pictured at Fig. 5.6 — where the rectangle has to be replaced by a diagram (called a tangle) with three arcs coming into it and three arcs existing out of it, we have $t^{-3}V_L(t) = V_{L'}(t)$. We show, in Fig. 5.7 the case of $\lambda = 1$.

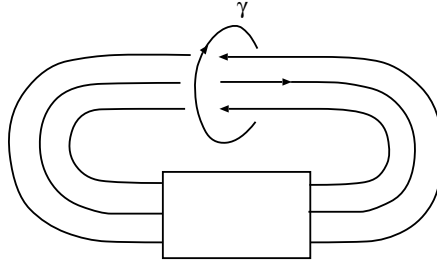


Fig. 5.7

Now one can use standard induction with respect to the number of crossings lying in the above rectangle and with respect to the number of bad crossings (for some choice of base points) in the rectangle to reduce the general situation in the rectangle to one of the six cases presented in Fig. 5.8. Subsequently, it is enough to apply steps (1) and (2) of the proof. The case of the second rectangle is illustrated in Figure 5.9.

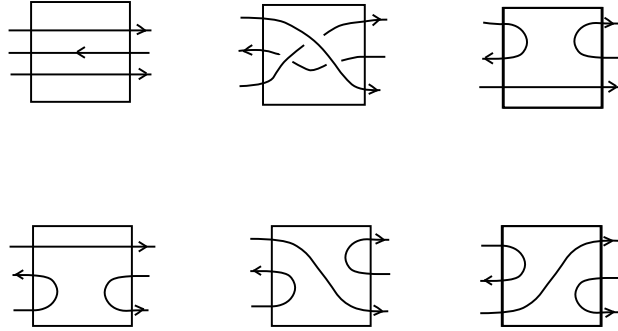


Fig. 5.8 six basic 3-tangles

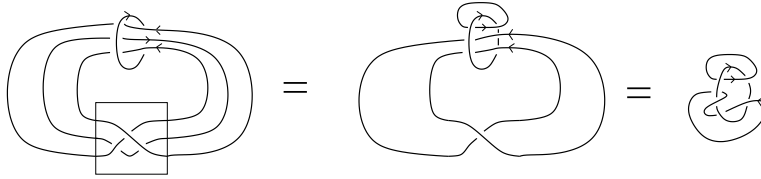


Fig. 5.9; Reduction in the case of the second rectangle

Similarly we deal with the case $n \leq 2$.

This completes the proof of Lemma III.5.15

The next step in the proof of Theorem III.5.13 is the so-called V_∞ -formula [Bi-2] in which we compare Jones polynomials of links L_+ , L_- , and L_∞ where L_∞ has one of possible two orientation which agree with orientation on L_+ for component not taking part in the modified crossing.

Lemma III.5.16 (Birman)

- (1) *The case $cp(L_+) = cp(L_0) - 1$ where $cp(L)$ denotes the number of components. Let L_∞ be given an orientation (as explained above) and let*

$\lambda = lk(L_i, L_0 - L_i)$ with L_i being the new component of L_0 , the orientation of which does not agree with the orientation of the corresponding component of L_∞ . Then

$$\sqrt{t}V_{L_+}(t) - \frac{1}{\sqrt{t}}V_{L_-}(t) = (\sqrt{t} - \frac{1}{\sqrt{t}})t^{-3\lambda}V_{L_\infty}(t).$$

- (2) The case $c(L_+) = c(L_0) + 1$: Let L_∞ be given any orientation (as explained above) and let $\lambda = lk(L_i, L_+ - L_i)$ with L_i being the component of L_+ , the orientation of which does not agree with the orientation of the corresponding component of L_∞ . Then:

$$\sqrt{t}V_{L_+}(t) - \frac{1}{\sqrt{t}}V_{L_-}(t) = (\sqrt{t} - \frac{1}{\sqrt{t}})t^{-3(\lambda-\frac{1}{2})}V_{L_\infty}(t).$$

Proof.

1. $cp(L_+) = cp(L_0) - 1$.

Let us consider a diagram X with two crossings, p and q (Fig. 5.10), such that $L_0 = X_-^{p\ q}_+$, $L_+ = X_+^{p\ q}_0$ and $L_- = X_-^{p\ q}_0$. Considering the crossing q we obtain:

- 2.

$$(a) \quad -tV_{L_0}(t) + \frac{1}{t}V_X(t) = (\sqrt{t} - \frac{1}{t})V_{L_-}(t).$$

Now let us change the orientation of the component of L_0 which exists at the upper right-hand corner (Fig. 5.10) and call the resulting link L'_0 . Similarly, we change X to X' (Fig. 5.10).

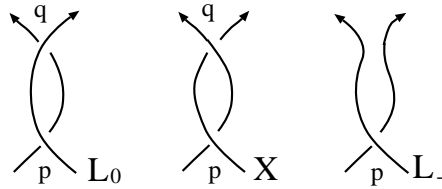


Fig. 5.10

Now let us choose the orientation of $L_\infty = X_-^{p\ q}_\infty = X_\infty^{p\ q}_- = X_+^{p\ q}_0$ so that it agrees with the orientation of L'_0 (Fig. 5.11).

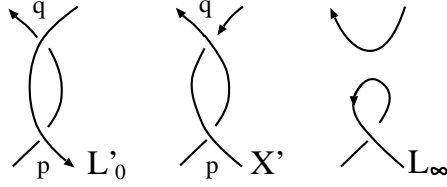


Fig. 5.11

Because of the diagrams at Fig. 5.11, considering the crossing q , we get $-tV_{X'}(t) + \frac{1}{t}V_{L'_0}(t) = (\sqrt{t} - \frac{1}{\sqrt{t}})V_{L_\infty}(t)$. Moreover, because of Lemma III.5.15, we have $V_{L'_0}(t) = t^{3\lambda}V_{L_0}(t)$ and $V_{X'}(t) = t^{3(\lambda-1)}V_X(t)$ and it follows that

$$(b) \quad \frac{1}{t}V_{L_0}(t) - \frac{1}{t^2}V_X(t) = (\sqrt{t} - \frac{1}{\sqrt{t}})t^{-3\lambda}V_{L_\infty}(t).$$

The triple consisting of L_+ , L_- and L_0 gives us the equation

$$(c) \quad -tV_{L_+}(t) + \frac{1}{t}V_{L_-}(t) = (\sqrt{t} - \frac{1}{\sqrt{t}})V_{L_0}(t).$$

The equation $(b) + \frac{1}{t}(a) - \frac{1}{\sqrt{t}}(c)$ (i.e. the sum of the above equations with the respective coefficients) is the V_∞ -formula we looked for.

$$3. \quad cp(L_+) = cp(L_0) + 1.$$

Let L'_+ be a link obtained from L_+ by changing of the orientation of L_i , and L'_- is obtained from L_- in a similar manner. Now the smoothing of L'_+ is exactly the link $L'_0 = L_\infty$. Let us apply the defining equation of the Jones polynomial for the triple L'_- , L'_+ and L'_0 . As the result we get:

$$-tV_{L'_-}(t) + \frac{1}{t}V_{L'_+}(t) = (\sqrt{t} - \frac{1}{\sqrt{t}})V_{L'_0}(t),$$

further, by Lemma III.5.15, we have $V_{L'_+}(t) = t^{3\lambda}V_{L_+}(t)$ and $V_{L'_-}(t) = t^{3(\lambda-1)}V_{L_-}(t)$. Therefore

$$(d) \quad -\frac{1}{t^2}V_{L_-}(t) + \frac{1}{t}V_{L_+}(t) = (\sqrt{t} - \frac{1}{\sqrt{t}})t^{-3\lambda}V_{L_\infty}(t),$$

which completes the proof of the part (2) and of the whole Lemma III.5.16.

Now we can conclude the proof of Theorem III.5.13. As before, we have to consider two cases:

$$(1) \quad pc(L_+) = c(L_0) - 1.$$

Let us consider the formula defining the Jones polynomial and also the V_∞ -formula from Lemma III.5.16. We get:

$$-tV_{L_+}(t) + \frac{1}{t}V_{L_-}(t) = (\sqrt{t} - \frac{1}{\sqrt{t}})V_{L_0}(t)$$

and

$$\sqrt{t}V_{L_+}(t) - \frac{1}{\sqrt{t}}V_{L_-}(t) = (\sqrt{t} - \frac{1}{\sqrt{t}})t^{-3\lambda}V_{L_\infty}(t).$$

Now, it is enough to add the above equations to get the first formula of Lemma III.5.14(2) for $a = t^{\frac{4}{3}}$ and $z = -(t^{-\frac{1}{4}} + t^{\frac{1}{4}})$.

$$(2) \quad cp(L_+) = cp(L_0) + 1.$$

Exactly the same argument as in the first case yields the second formula of Lemma III.5.14(2).

This completes the proof of Theorem III.5.13.

The proof of Theorem III.5.13 which we presented above reflects the history of understanding the Jones polynomial as in [Jo-1], [Bi-2], [L-M-3] and [Li-2]. The proof can be actually shortened if, instead of $V_L(t)$, we consider the polynomial $t^{\frac{3}{4}w(L)}V_L(t)$ which is an invariant of regular isotopy.

Earlier, in Theorems 5.9, 5.13 and Lemmas 5.10, 5.14, we have already presented some basic properties of the Kauffman polynomial the result below is another interesting and unexpected property.

While working on t_4 moves I have suggested (and partially proved in April of 1986) that $F_K(a, -a - a^{-1}) = 1$ if K is a knot, [Mo-4]. Lickorish and Millett proved this conjecture and at the same time they found a formula for the Kauffman polynomial of an arbitrary link and for $z = -a - a^{-1}$ [L-M-4] (c.f. [Lip-1]). The formula for $F_L(a, -a - a^{-1})$ was found independently, in a more general context, by Turaev [Tu-1]. Below we present a proof of the result, which needs very little of computations and no previous knowledge (guess) of the formula. It is based on the idea of presenting an unoriented link as a sum of all oriented links which are obtained from the given one by setting all possible orientations [P-7].¹⁹

¹⁹This idea was presented for the first time in [Gol] and it was credited to Dennis Johnson (compare also with [Tu-2, H-P-2]).

Theorem III.5.17 *For any oriented link L we have the following formula*

$$F_L(a, -a - a^{-1}) = ((-1)^{cp(L)-1}/2) \sum_{S \subset L} a^{-4lk(S, L-S)},$$

where the summation is over all sublinks S of the link L (including $S = \emptyset$), and $cp(L)$, $lk(S, L - S)$ denote, respectively, the number of components of the link L and global linking number of S and $L - S$.

Proof: Let us consider the following very simple invariant of oriented diagrams $g(D) := (-1)^{cp(D)} a^{w(D)} \in \mathbb{Z}[a^{\pm 1}]$. Let us note that $g(D_+) = -ag(D_0) = a^2g(D_-)$. Moreover, g is an invariant of regular isotopy and the first Reidemeister move changes g by $a^{\pm 1}$. Now, let us extend the definition of g for unoriented diagrams. To avoid ambiguity the extended invariant is denoted by G . If D is an unoriented diagram then the invariant G is given by the following formula

$$G(D) = \sum_{D' \in OR(D)} g(D')$$

where $OR(D)$ is the set of all orientations of the diagram D . We see immediately that G is an invariant of regular isotopy of unoriented links. Besides, G has the following properties:

- (1) $G(R_{1+}(D)) = aG(D)$ and $G(R_{1-}(D)) = a^{-1}G(D)$, where R_{1+} (respectively, R_{1-}) is the first Reidemeister move which introduces a positive (respectively, negative) twist,
- (2) $G(D_+) = -a(G(D_0) + G(D_\infty))$ if the crossing of D_+ is a positive selfintersection.
- (3) $G(D_-) = -a^{-1}(G(D_0) + G(D_\infty))$ if the crossing of D_- is a negative selfintersection.
- (4) $G(D_+) = -aG(D_0) - a^{-1}G(D_\infty)$ in the case of a mixed crossing (the meaning of D_+ (resp. D_-) for a mixed crossing is chosen as follows: We consider an orientation on $D_{\pm 1}$ in such a way that an oriented smoothing gives D_0 ; D_+ (resp. D_-) means that the smoothed crossing is positive (resp. negative).
- (5) $G(D_-) = -a^{-1}G(D_0) - aG(D_\infty)$ in the case of a mixed crossing with D_- explained above.

Formulas (2) and (3) follow from the observation that orientations of D_0 are in bijection with orientations of D_\pm and D_∞ (taken together). Similarly (4) and (5) follow from the observation that orientations of D_\pm are in bijections with orientations of D_0 and D_∞ .

Now, adding side-to-side equations (2) and (3) as well as (4) and (5) we obtain the following formula for an arbitrary crossing of an unoriented diagram D :

$$(6) \quad G(D_+) + G(D_-) = (-a - a^{-1})(G(D_0) + G(D_\infty)).$$

The properties (1) and (6) are also conditions defining the Kauffman polynomial of unoriented links, $\Lambda_D(a, z)$ for $z = -a - a^{-1}$. We have yet to compare the initial conditions. Namely, for a trivial circle T_1 we have $G(T_1) = -2 = -2\Lambda_{T_1}$. Therefore $G(D) = -2\Lambda_D(a, -a - a^{-1})$ and further for a diagram of an oriented link we have:

$$(7) \quad F_{D'}(a, -a - a^{-1}) = a^{-w(D')} \Lambda_D(a, -a - a^{-1}) = -\frac{1}{2} a^{-w(D')} G(D),$$

where D is an unoriented diagram obtained from D' by forgetting about its orientation. By definition we have

$$(8) \quad G(D) = \sum_{D' \in OR(D)} g(D') = (-1)^{cp(D)} \sum_{D' \in OR(D)} a^{w(D')}.$$

Now, if L is an oriented diagram of a link and S is its sublink, and L_S is obtained from L by changing the orientation of S , then $w(L_S) - w(L) = 2lk(L_S) - 2lk(L) = -4lk(S, L - S)$. This equality and the formula (8) can be applied for any orientation D'' of the unoriented diagram D to get:

$$(9) \quad a^{-w(D'')} G(D) = (-1)^{cp(L)} \sum_{S \subset D''} a^{-4lk(S, D'' - S)}.$$

Finally, formulas (7) and (9) give

$$F_{D'}(a, -a - a^{-1}) = -\frac{1}{2} a^{-w(D')} G(D) = \frac{1}{2} (-1)^{w(D')-1} \sum_{S \subset D'} a^{-4lk(S, D' - S)},$$

which is the formula of Theorem III.5.17.

□

Exercise III.5.18 *Show that for unoriented diagrams D_+ , D_- , D_0 , and D_∞ with conventions like in (2)-(5) of the proof of Theorem III.5.17.*

(a) $G(D_+) - G(D_-) = (a^{-1} - a)(G(D_0) + G(D_\infty))$ in the case of selfcrossing,

- (b) $a^{-1}G(D_+) - aG(D_-) = 0$ in the case of selfcrossing,
- (c) $G(D_+) - G(D_-) = (a^{-1} - a)(G(D_0) - G(D_\infty))$ in the case of mixed crossing,
- (d) $aG(D_+) - a^{-1}G(D_-) = (a^{-2} - a^2)G(D_0)$, in the case of mixed crossing.

Exercise III.5.19 Let $F^*(a, z)$ be the Dubrovnik version of the Kauffman polynomial²⁰. The polynomial F^* satisfies a recursive condition

$$a^{w(D_+)}F_{D_+}^* - a^{w(D_-)}F_{D_-}^* = z(a^{w(D_0)}F_{D_0}^* - a^{w(D_\infty)}F_{D_\infty}^*).$$

Find out the value of $F_L^*(ia, i(a - a^{-1}))$. Find a general relation between $F^*(a, z)$ and $F(a, z)$ ([Li-3]).

Exercise III.5.20 The invariant g of oriented diagrams can be extended or, more precisely, quantized (using popular parlance). The quantization leads to the following invariant of oriented links which generalizes the polynomial $F_L(a, -a - a^{-1})$, namely:

$$\hat{G}(a, x) = \sum_{S \subset L} x^{cp(S)} a^{-4lk(S, L-S)}.$$

Show that $\hat{G}(a, x)$ is sometimes a better invariant than the original Kauffman polynomial. For example, it distinguishes 4-component links which are presented on Fig. 3.1. Is it true that $\hat{G}(a, x)$ can distinguish 3-component links which are not distinguished by the Kauffman polynomial?

Problem III.5.21 By the determinant of an oriented link we understand a numerical invariant obtained by an appropriate evaluation of an invariant polynomial (i.e. either Conway, or Jones, or Jones-Conway, or Kauffman polynomial) of the link (see Chapter IV for detailed discussion of link determinant). Namely, $D_L = \Delta_L(-2i) = P_L(\frac{1}{2}i, -\frac{1}{2}i) = V_L(-1)$ (more precisely $\sqrt{t} = -i = F((-i)^{\frac{3}{2}}, -((-i)^{-\frac{1}{2}} + (-i)^{\frac{1}{2}}))$). Now, if we use the variant of the Homflypt polynomial which is used in the definition of the supersignature then $D_L = r(\frac{1}{2}, \frac{1}{2})$. The signature $\sigma(L)$ is a supersignature modeled on the determinant of the link, $\sigma(L) = \sigma_{\frac{1}{2}, \frac{1}{2}}(L)$. In the Chapter IV we will show that the signature has properties similar to Jones Lemma III.5.15 (also compare with Lemma III.5.10). Namely, if L_i is a component of an oriented

²⁰Kauffman described the polynomial F^* on a postcard to Lickorish sent from Dubrovnik in September 1985. He expected that this was a new polynomial invariant of links, independent from F .

link L and $\lambda = \text{lk}(L_i, L - L_i)$ and, moreover, if L' is obtained from L by changing the orientation of L , then $\sigma(L') = \sigma(L) + 2\lambda$. This can not be an accident and a similar formula should be true for other supersignatures (especially these modeled on the Jones polynomial). We leave this question as a research problem for the reader.

Below we introduce a new equivalence relation similar to \sim_c (Conway or skein equivalence). The new relation is motivated by a four term relation of Kauffman polynomial, and bound the applicability of Kauffman method like \sim_c bounds applicability of Conway type invariants.

Definition III.5.22 *Let S be the set of partially oriented diagrams (i.e. diagrams which may contain some oriented and some unoriented components) modulo regular isotopy equivalence. The Kauffman equivalence relation \sim_K is the smallest equivalence relation on the set S which satisfies the following condition:*

If L'_1 (resp. L'_2) is a diagram representing an element L_1 (resp. L_2) of S which contains a crossing p_1 (resp. p_2) and moreover:

- (i) $(L'_1)^{p_1}_{-} \text{sgn } p_1 \sim_K (L'_2)^{p_2}_{-} \text{sgn } p_2$ where $L^p_{-} \text{sgn } p$ denotes the link obtained from L by interchanging overcrossing and undercrossing in p (this does not depend on whether and how L is oriented),
- (ii) $(L'_1)^{p_1}_0 \sim_K (L'_2)^{p_2}_0$ and $(L'_1)^{p_1}_\infty \sim_K (L'_2)^{p_2}_\infty$ provided that p_1 is a crossing of oriented components of L'_1 or p_1 is a selfcrossing of a component of L'_1 (in the latter case $(L'_1)^{p_1}_0$ and $(L'_1)^{p_1}_\infty$ are well defined independently of the orientation of the component),
- (iii) $\{(L'_1)^{p_1}_0, (L'_1)^{p_1}_\infty\} = \{(L'_2)^{p_2}_0, (L'_2)^{p_2}_\infty\}$ (equality of unordered pairs of links up to Kauffman equivalence) if at least one of two components of L'_1 , which meet in p_1 , is unoriented,

then $L_1 \sim_K L_2$.

Let us note that the crossings p_1 and p_2 satisfy the conditions listed in Definition III.5.22 only if they have similar properties. For example, if two components of L_1 meeting in p_1 are oriented then components of L_2 meeting in p_2 are oriented as well and moreover $\text{sgn } p_1 = \text{sgn } p_2$.

Exercise III.5.23 *Formulate conditions for oriented diagrams L_1 and L_2 , such regular isotopy implies regular isotopy (i.e. the situation when the diagrams which are not regularly isotopic are also not isotopic).*

Hint. Consider the writhe number $w(\)$ of any component of the diagram and another invariant of regular isotopy of oriented diagrams, which we define as follows. Let us consider a diagram L . After smoothening of all crossings of L we obtain a set of oriented circles on the plane. Some of them are oriented positively (\bigcirc) some negatively (\bigcirc); the invariant in question is the result of the subtraction of the number of “negative” circles from the number of “positive” circles. This invariant is often called the Whitney degree or rotation number (and defined for any curve or system of curves immersed in plane).

Remark III.5.24 *In applications we do not need always such a subtle differentiation between isotopy and regular isotopy. This is because of the following facts:*

- Exercise III.5.25** 1. *Prove that if L_2 is obtained from L_1 by the first weakened Reidemeister move ($R_{0.5}^{\pm 1}$) then $L_1 \sim_K L_2$.*
2. *Prove that if L_1 is isotopic with L_2 and $w(L_1) = w(L_2)$ then $L_1 \sim_K L_2$ (compare Lemma III.5.6).*

Exercise III.5.26 *Prove that if an oriented diagram L_1 is a mutant of another oriented diagram L_2 then $L_1 \sim_K L_2$.*

Now we show how invariants of links obtained by the Kauffman approach can be described by one algebraic structure — similarly as Conway type invariants are described by the notion of Conway algebra. We will also construct an example of a polynomial invariant of oriented links which is a generalization of both, Jones-Conway and Kaufman polynomials (however, this invariant does not bring anything more than these two polynomial together). We will proceed similarly as in the case of Conway algebra, with the exception that we will consider diagrams up to regular isotopy. Since there is no need to distinguish the crossings of $+$ type from the crossings of $-$ type, there will be only one operation recovering the value of the invariant for L_+ (resp. L_-) from its values for L_- , L_0 and L_∞ (resp. L_+ , L_0 and L_∞). There is yet another important remark: if a link L_+ is modified to L_∞ then the new component of L_∞ can not be assigned any natural orientation, we could have, however, considered the partially oriented link L . We do not do it for the practical reason, since in such a case we would like to use a method similar to the one used in the proof of Theorem III.1.2 and for this we need the following equality: $L_\varepsilon^p q_\infty = L_\infty^q p_\varepsilon$, where $\varepsilon = +, -$ or

0. This means that we would like the result of operations performed on two crossings to be independent of the order of these operations. But this is not always the case, as is presented in Fig. 5.12.

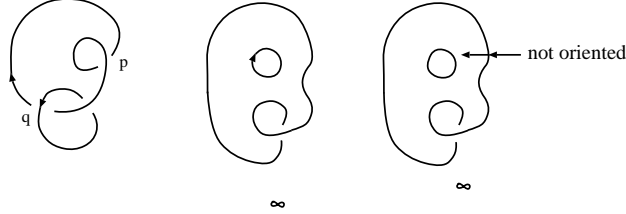


Fig. 5.12; diagrams L , $L_{0,\infty}^{pq}$ and $L_{\infty,0}^{qp}$ (last two diagrams are not the same as partially oriented diagrams)

Therefore we will restrict the consideration to the case of oriented and unoriented link, that is we will either assume that all components are oriented or none of them is oriented. In the latter case we will not distinguish L_0 from L_∞ .

Let us consider the following general situation. Suppose that we are given an abstract algebra \mathcal{A} with two universes, A and A' , and with a countable (possibly finite) number of 0-argument operations in A and A' , parameterized by two sequences: $\{a_{ij}\}_{i \in N, j \in \mathbb{Z}}$ and $\{a'_{ij}\}_{i \in N, j \in \mathbb{Z}}$, respectively. Moreover, \mathcal{A} is assumed to have two 3-argument operations, namely $*$: $A \times A \times A' \rightarrow A$ and $*'$: $A' \times A' \times A' \rightarrow A'$, and also 1-argument operation $\phi : A \rightarrow A'$. We would like to construct invariants of classes of regular isotopy of oriented and unoriented diagrams, which satisfy the following conditions:

1. If L is an oriented diagram then the invariant w is in A , i.e. $w_L \in A$ and if L' is unoriented then $w_{L'} \in A'$.
2. If L' is an unoriented diagram obtained by forgetting the orientation of an oriented diagram L then $w_{L'} = \phi(w_L)$.
3. $w_{T_{ij}} = a_{ij}$ where T_{ij} denotes an oriented diagram of a trivial link with i components and the write number $w(T_{ij}) = j$.
4. $w_{T'_{ij}} = a'_{ij}$.
5. $w_{L^p} = w_{L_-^p} *_{\text{sgn } p} (w_{L_0^p}, w_{L_\infty^p})$ where L_-^p denotes a diagram obtained from L by exchanging the tunnel to the bridge at the crossing p .

Definition III.5.27 We say that $\mathcal{A} = \{A, A', \{a_{ij}\}, \{a'_{ij}\}, *, *', \phi\}$ is a Kauffman algebra if the following conditions are satisfied:

$$K1 \quad \phi(a_{ij}) = a'_{ij}$$

$$K2 \quad \phi(a * (b, c)) = \phi(a) *' (\phi(b), c) \text{ where the operation } * \text{ on } (a, b, c) \text{ is denoted by } a * (b, c) \text{ and similarly for the operation } *'.$$

$$K3 \quad a_{i,j-1} * (a_{i+1,j}, a'_{i,j}) = a_{i,j+1}.$$

$$K4 \quad (a * (b, c)) * (d * (e, f), g *' (h, i)) = (a * (d, g) * (b * (e, h)), c *' (f, i)).$$

$$K5 \quad (a * (b, c)) * (b, c) = a.$$

$$K6 \quad a *' (b, c) = a *' (c, b)$$

Theorem III.5.28 For a given Kauffman algebra \mathcal{A} there exists a uniquely determined invariant of regular isotopy w which associates an element w_L from A to any oriented diagram L and an element $w_{L'}$ from A' to any un-oriented diagram L' . Moreover, the invariant w satisfies the following conditions:

1. $w_{T_{i,j}} = a_{i,j}$
2. $w'_{L'} = \phi(w_L)$, where L is an oriented diagram and L' is obtained from L by forgetting its orientation.
3. $w_{L^p} = w_{L_-^p} \text{sgn}_p * (w_{L_0^p}, w_{L_\infty^p})$.

The proof of Theorem III.5.28 is similar to that of Theorem III.1.2, therefore we present only these parts of the argument in which some differences occur.

For any diagram (oriented or not) we can build a resolving tree such that three edges descend from any vertex which is not a leaf (Fig. 5.13) and there are descending diagrams at the leaves (the diagrams are descending for some choice of base points and orientation — in case of unoriented diagrams).

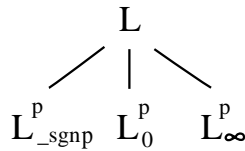


Fig. 5.13

Such a tree can be used to compute the value of the invariant at the root diagram. We begin by constructing the invariant w for diagrams and subsequently we will show that it is not changed by Reidemeister moves $R_{0.5}^{\pm 1}$, $R_2^{\pm 1}$ and $R_3^{\pm 1}$. We use induction with respect to the number of crossings in the diagram, denoted now, as in the proof of Theorem by $\text{cr}(L)$. For any $k \geq 0$ we define a function w_k which associates an element from A (reps. A') to any oriented (resp. unoriented) diagram with at most k crossings. The invariant w is then defined as $w_L = w_k(L)$ where $k \geq \text{cr}(L)$. Similarly as in the proof of Theorem III.1.2 we set $w_0(L) = a_{n,0}$ if L is a trivial oriented link with n components and we put $w_0(L') = a'_{n,0}$ if L' is obtained from L by forgetting its orientation. Next we formulate the Main Inductive Hypothesis, or MIH. We assume that we have already defined a function w_k which assigns an element from A (resp. A') to any diagram L with $\text{cr}(L) \leq k$ and the function w_k has the following properties:

Property III.5.29 *Suppose that $U_{n,j}$ is an oriented diagram which is descending for some choice of base points; moreover $U_{n,j}$ has n components, $\text{cr}(U_{n,j}) \leq k$ and $\text{Tait}(U_{n,j}) = j$ (we use $\text{Tait}(D)$ in place of $w(D)$ so not to mix the writhe number of an oriented diagram with our invariant w). Then $w_k(U_{n,j}) = a_{n,j}$. Similarly, if $U'_{n,j}$ is obtained from $U_{n,j}$ by forgetting its orientation then $w_k(U'_{n,j}) = a'_{n,j}$.*

Property III.5.30 $w_k(L) = w_k(L_-^p \text{sgn } p) * (w_k(L_0^p), w_k(L_\infty^p))$ if L is an oriented diagram and $w_k(L) = w_k(L_-^p \text{sgn } p) *' (w_k(L_0^p), w_k(L_\infty^p))$ if L is unoriented.

Property III.5.31 $w_k(L) = w_k(R(L))$ where R is a Reidemeister move of one of the following types: $R_{0.5}^{\pm 1}$, $R_2^{\pm 1}$, $R_3^{\pm 1}$ and $\text{cr}(R(L)) \leq k$.

Next we want to prove Main Inductive Step or MIS, that is, we want to define the function w_{k+1} with appropriate properties, which is defined for diagrams with at most $k+1$ crossings. Similarly as in the case of Theorem III.1.2, this will complete the proof of Theorem III.5.28. We begin the proof of MIS, similarly as for III.1.2 by defining a function w_b which for a diagram L with $\text{cr}(L) = k+1$ depends on the choice of the base points $b = (b_1, b_2, \dots, b_n)$ and on the choice of the orientation, if L is unoriented. We define the function w_b by induction with respect to the number of bad crossings, $b(L)$, and we apply the formula III.5.30 for the first bad crossing — similarly as in the proof of Theorem III.1.2). Subsequently, we prove

that formula III.5.30 is satisfied for any crossing. The argument is similar to the appropriate one in the course of the proof of Theorem III.1.2, only the conditions C3–C5 are replaced by the condition K4.

The next step of the proof is to show that w_b does not depend on the choice of base points (assuming the given order and orientation of components). Again, we deal with this problem similarly as in III.1.2 and we choose base points b and $b' = (b_1, b_2, \dots, b'_i, \dots, b_n)$, where b_i and b'_i are on the opposite sides of a crossing of the component L_i . Here we apply induction with respect to $B(L) = \max(b(L), b'(L))$. If $B(L) = 0$ then L is a descending diagram with respect to both b and b' and therefore $w_b(L) = w_{b'}(L) = a_{n, \text{Tail}(L)}$. If $B(L) = b(L) = b'(L) = 1$ then L has a common bad crossing for both b and b' , hence we can make the inductive step using this crossing (and also III.5.30).

Thus, we are left with the case $B(L) = 1$ and $b(L) \neq b'(L)$. The argument here is a little more complicated than the argument in the respective part of the proof of III.1.2. Namely: let p be the bad crossing of L with respect to either b or b' , then p is a selfintersection of the component $L_i \subset L$. For simplicity, let us assume that the diagram L is oriented and $b(L) = 1$, $b'(L) = 0$ and $\text{sgn } p = +$. Then L is descending with respect to b' , therefore

$$w_{b'}(L) = a_{n, \text{Tail}(L)}.$$

Because of the property III.5.30 we have

$$w_b(L) = w_b(L_-^p) * (w_b(L_0^p), w_{b'}(L_\infty^p)).$$

Moreover $b(L_-^p) = 0$ hence $w_b(L_-^p) = a_{n, \text{Tail}(L)-2}$. The diagram L_0^p is descending for some choice of base points it has k crossings and $n+1$ components, hence $w_b(L_0^p) = w_k(L_0^p) = a_{n+1, \text{Tail}(L)-1}$. Now, in order to apply the axiom K3 to get equality $w_b(L) = a_{n, \text{Tail}(L)} = w_{b'}(L)$ we have to prove that $w_b(L_\infty^p) = a_{n, \text{Tail}(L)-1}$. However, this is not immediate. Namely, L_∞^p does not have to be descending with respect to any choice of base points or orientation. We can use the fact that the diagram L_∞^p has only k crossings and it consists of a descending part and an ascending part, and these parts may be put on different levels (Fig. 5.14 illustrates the situation). Now to prove that the value of the invariant w for L_∞^p is equal $a_{n, \text{Tail}(L)-1}$ we have to apply the following trick:

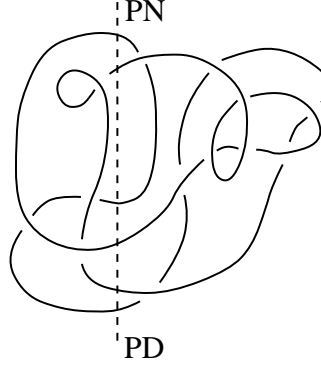


Fig. 5.14

Let us rotate the ascending part of the diagram of L_∞^p by 180° with respect to the vertical (N-S) axis and at the same time let us change the orientation of this part to the opposite (with respect to the given orientation of L_∞^p) — this way we make some kind of a mutation. The resulting diagram, call it \tilde{L} , is descending.

Therefore $w_k(\tilde{L}) = a'_{n, \text{Tait}(L)-1}$. On the other hand, we can build the same resolving tree for L_∞^p and \tilde{L} such that all vertices of the tree correspond to diagrams with at most k crossings (there is a complete analogy with the operation of mutation). Now, because of MIH we have $w_k(L_\infty^p) = w_k(\tilde{L}) = a'_{n, \text{Tait}(L)-1}$, which completes the proof, c.f. [B-L-M].

The rest of the proof of Theorem III.5.28 is almost the repetition of the respective part of the argument in the proof of Theorem III.1.2. We change the Reidemeister move $R_1^{\pm 1}$ to $R_{0.5}^{\pm 1}$. The proof of the independence of the order of components in w_b is the same as in III.1.2 because in Lemma III.2.15 Reidemeister move $R_1^{\pm 1}$ can be replaced by $R_{0.5}^{\pm 1}$. If L is not oriented then applying Lemma III.2.15 we show that the definition of w_b does not depend on the choice of the orientation that we have made.

This completes the proof of Theorem III.5.28.

Example III.5.32 (Jones-Conway-Kauffman polynomial)

Let us consider the following algebra \mathcal{A} :

$$A = \mathbb{Z}[a^{\pm 1}, t^{\pm 1}, z], \quad A' = \mathbb{Z}[a^{\pm 1}, t^{\pm 1}],$$

$$a_{i,j} = \left(\frac{a^{-1}+a}{t}\right)^{i-1} \left(1 - \frac{z}{t}\right) a^j + \frac{z}{t} \left(\frac{a^{-1}+a}{t} - 1\right)^{i-1} a^j,$$

$$a'_{i,j} = \left(\frac{a^{-1}+a}{t} - 1\right)^{i-1} a^j,$$

moreover $b * (c, d)$ is defined by the equation $b * (c, d) + b = tc + zd$ and $b *' (c, d)$ is defined by the equation $b *' (c, d) + b = tc + td$; the 1-argument operation $\phi : A \rightarrow A'$ is defined on generators $\phi(a) = a, \phi(t) = t, \phi(z) = t$.

Now we will check that \mathcal{A} is a Kauffman algebra. The conditions K1, K2, K5 and K6 follow immediately by the definition of \mathcal{A} . The condition K3 follows by the equality:

$$\begin{aligned}
& \left(\frac{a^{-1}+a}{t}\right)^{i-1} \left(1 - \frac{z}{t}\right) a^{j+1} & + \\
+ & \frac{z}{t} \left(\frac{a^{-1}+a}{t} - 1\right)^{i-1} a^{j+1} & + \\
+ & \left(\frac{a^{-1}+a}{t}\right)^{i-1} \left(1 - \frac{z}{t}\right) a^{j-1} & + \\
+ & \frac{z}{t} \left(\frac{a^{-1}+a}{t} - 1\right)^{i-1} a^{j-1} & = \\
& t \left(\left(\frac{a^{-1}+a}{t}\right)^i \left(1 - \frac{z}{t}\right) a^j + \frac{z}{t} \left(\frac{a^{-1}+a}{t} - 1\right)^i a^j\right) & + \\
+ & z \left(\frac{a^{-1}+a}{t} - 1\right)^{i-1} a^j
\end{aligned}$$

It remains to check the condition K4:

$$\begin{aligned}
& (a * (b, c)) * (d * (e, f), g *' (h, i)) & = \\
= & -(a * (b, c) + t(d * (e, f)) + z(g *' (h, i))) & = \\
= & -(-a + tb + zc) + t(-d + te + zf) + z(-g + th + ti) & = \\
= & a - tb - zc - td + t^2e + tzf - zg + zth + zti & = \\
= & (a * (d, g)) * (b * (e, h), c *' (f, i)).
\end{aligned}$$

The polynomial invariant of regular isotopy of oriented or unoriented diagrams defined by the algebra \mathcal{A} is called Jones-Conway-Kauffman polynomial and it is denoted by $J_L(a, t, z)$. It can be modified to an invariant of oriented links by setting $\tilde{J}_L(a, t, z) = J_L(a, t, z) a^{-\text{Tail}(L)}$.

Example III.5.33 Let us compute the Jones-Conway-Kauffman polynomial for a right-hand-side trefoil knot which is presented on Fig. 5.15. If we apply the resolving tree presented on Fig. 5.15 then we will get the following relation in any Kauffman algebra: $w_L = a_{1,1} * (a_{2,0} * (a_{1,1}, a'_{1,-1}), a'_{1,-2})$.

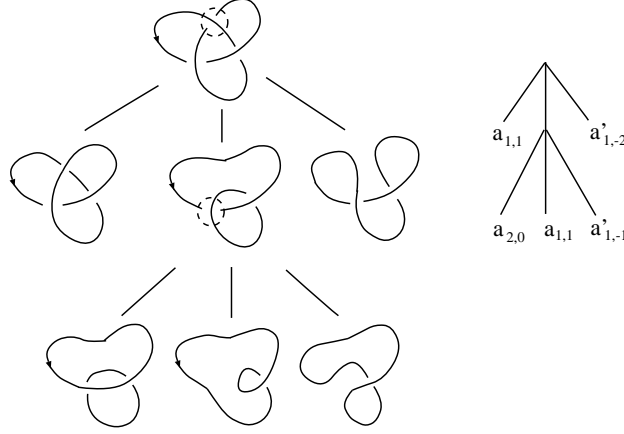


Fig. 5.15

Thus we get:

$$\begin{aligned}
 J_L(a, t, z) &= -a + t\left(\frac{a^{-1}+a-z}{t} + ta + za^{-1}\right) + za^{-2} = \\
 &\quad -a^{-1} - 2a + t^2a + z(1 + a^{-2} + ta^{-1}). \\
 \tilde{J}_L(a, t, z) &= -a^{-4} - 2a^{-2} + t^2a^{-2} + z(a^{-3} + a^{-5} + ta^{-4}).
 \end{aligned}$$

Lemma III.5.34 (1) $J_L(a, t, z) = J_L(a, t, 0) + z\left(\frac{J_L(a, t, t) - J_L(a, t, 0)}{t}\right)$.

(2) $\tilde{J}(a, t, 0) = P\left(\frac{a}{t}, \frac{1}{at}\right)$, thus \tilde{J} is a version of the Homflypt polynomial.

(3) $J_L(a, t, t) = G_L(a, t)$, so J is the Kauffman polynomial for regular isotopy.

Proof. The property (1) is obviously true for diagrams representing trivial links. Next we apply induction with respect to the length of the resolving tree of the diagram.

To prove (2) and (3) it is enough to check the initial conditions and to compare the operations $*$ which are used in the respective definitions of polynomial invariants.

Therefore the Jones-Conway-Kauffman polynomial is equivalent to Jones-Conway (Homflypt) and Kauffman polynomials taken together. In this context, there is a remarkable similarity with the situation of Exercise III.3.43.

Lemma III.5.35 *The invariants defined by Kauffman algebras are invariants of \sim_K equivalence of oriented or unoriented diagrams.*

The proof is immediate.

Remark III.5.36 *The theory of invariants defined by Kauffman algebras can be developed in parallel to the theory of invariants given by Conway algebras. In particular:*

1. *One may look for a pair of involutions τ on A and τ' on A' such that $\tau(a_{i,j}) = a_{i,-j}$, $\phi(\tau(w)) = \tau'(\phi(w))$ and $\tau(a * (b, c)) = \tau(a) * (\tau(b), \tau'(c))$. Then $A_{\overline{L}} = \tau(A_L)$ where A_L is an invariant of an oriented diagram L and \overline{L} is the mirror image of L (compare with Lemma III.3.22).*
2. *It is possible to build a universal Kauffman algebra (the elements of which are terms) and to show that for such a universal Kauffman algebra the involutions τ and τ' exist.*
3. *It is reasonable to look for an operation $o : A \times A \times A' \rightarrow A$ which, for given oriented diagrams, would recover the value of the invariant for L_0 from its values for L_+ , L_- and L_∞ .*
4. *One may look for conditions for a Kauffman algebra which should provide a simple formula for the value of the associated invariant for connected sum and disjoint sum of diagrams.*
5. *One may look for conditions for a Kauffman algebra which would allow a simple modification of the associated invariant of regular isotopy to an invariant of ambient isotopy of diagrams. For example, if there exist bijections $\beta : A \rightarrow A$ and $\beta' : A' \rightarrow A'$ satisfying $\beta(a_{i,j}) = a_{i,j-1}$, $\phi(\beta(a)) = \beta'(\phi(a))$ and $\beta(a * (b, c)) = \beta(a) * (\beta(b), \beta'(c))$ then $\underbrace{\beta(\beta(\dots \beta(A_L) \dots))}_{\text{Tait}(L) \text{ times}}$ is an invariant of ambient isotopy.*
6. *One may consider geometrically sufficient partial Kauffman algebras (similarly to the case of Conway algebras, see Definition III.4.2) which would define invariants of regular isotopies of diagrams.*
7. *Similarly as in Example III.4.5, one may consider polynomials of infinite number of variables which generalize the Jones-Conway-Kauffman polynomial.*
8. *It is possible to show that the invariants defined by geometrically sufficient partial Kauffman algebras are preserved by mutations and, further, that mutations preserve classes of \sim_K equivalence of diagrams.*

We leave a possible extension of ideas presented above for the reader.

Many problems which we have formulated for invariants of Conway type can be extended for invariants obtained via the Kauffman method.

Problem III.5.37

1. *Do there exist two oriented diagrams which have the same Jones-Conway-Kauffman polynomial and which can be distinguished by another invariant obtained from some Kauffman algebra?*
2. *Do there exist two oriented diagrams which can not be distinguished by any invariant defined by a Kauffman algebra but they can be distinguished by invariants coming from a geometrically sufficient partial Kauffman algebra?*
3. *Do there exist two oriented diagrams which are not \sim_K equivalent and which can not be distinguished by any invariant defined by a geometrically sufficient partial Kauffman algebra?*
4. *If an oriented diagram of a knot, L , satisfies $L \sim_K \overline{L}$ is it true then that L is isotopic to either L or to $-\overline{L}$?*

It seems that a positive answer for (2) can be obtained by applying signature. The question (4) is a weakened version of Kauffman's conjecture III.5.11.

Exercise III.5.38 *Let us consider moves of diagrams: $R_{0,1}$ and $R_{0,2}$, as shown on Fig. 5.16. Let L be a diagram with k crossings. Prove that there exists a choice of base points b of L such that the descending diagram L^d , associated to L and b , can be modified to a diagram with fewer than k crossings via a sequence of moves consisting of $R_{0,1}$, $R_{0,2}$, $R_2^{\pm 1}$ and $R_3^{\pm 1}$, which do not increase the number of crossings.*

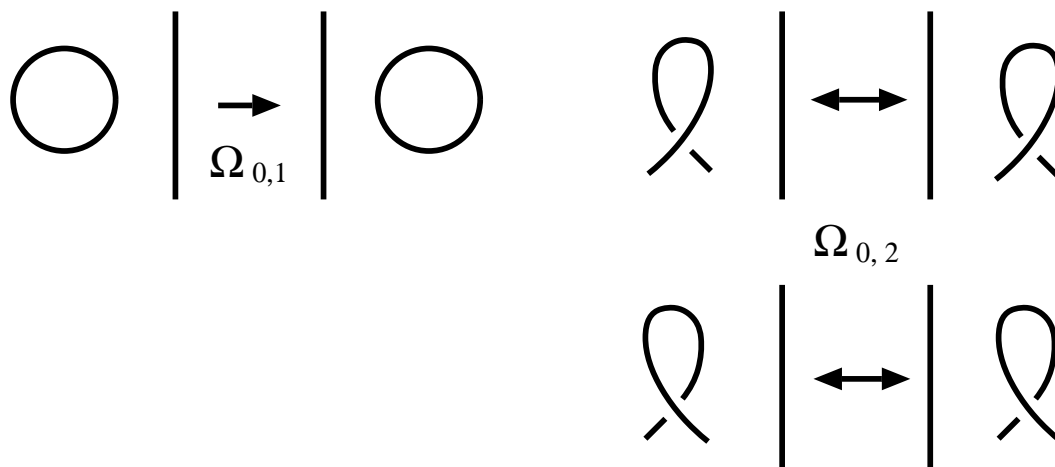


Fig. 5.16

Hint. See Lemma III.2.15.

Problem III.5.39 When we were defining invariants of diagrams via Kauffman algebras, or when we were defining the relation \sim_K , we had a problem with a natural orientation for the whole diagram L_∞^p . This was because the new component of L_∞^p was formed from the pieces of L which had opposite orientations. (Fig. 5.17).

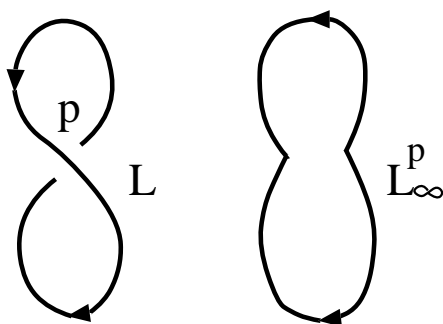


Fig. 5.17

Therefore, it seems to be reasonable to consider diagrams, the components of which can have different orientations, i.e. any component is divided into arcs and each arc has its orientation. Even a simple diagram (Fig. 5.18)

presents new difficulties which we have to consider (resolve the diagram first starting from p , and then from q). The author tried to compute the new invariant and his computations show that the problem is difficult but not hopeless. We leave it as a research problem for the reader.

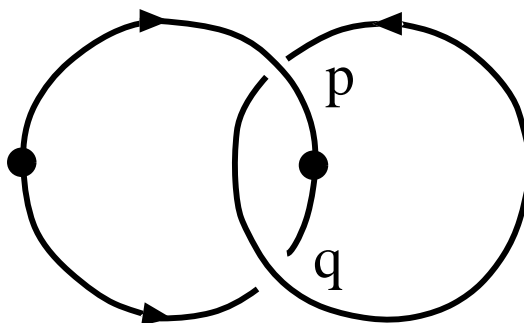


Fig. 5.18

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